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New upper bounds for the number of partitions into a given number of parts

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ABSTRACT

Binomial coefficients can be expressed in terms of multinomial coefficients as sums over integer partitions. This approach allows us to introduce new upper bounds for the number of partitions into a given number of parts.

1. Introduction

A composition of a positive integer n is a way of writing n as a sum of positive integers, i.e.,

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k.$$

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When the order of integers λ_i does not matter, the representation (1) is known as an integer partition [1] and can be rewritten as

$$n = t_1 + 2t_2 + \cdots + nt_n,$$

where each positive integer i appears t_i times in the partition. The number of parts of this partition is given by

$$t_1 + t_2 + \cdots + t_n = k.$$

The function giving the number of unrestricted partitions of n is usually denoted by $p(n)$. We consider $p(0) = 1$ and $p(n) = 0$ for any negative integer n. The number of partitions of n into k parts is denoted in this paper by $p(n,k)$. It is clear that

$$\sum_{k=1}^{n} p(n,k) = p(n).$$

A connection between binomial coefficients and multinomial coefficients is given by the following formula published by N.J. Fine [3, Ex. 5, p. 87]: for $n, k > 0$,

$$\sum_{\substack{t_1+2t_2+\cdots+nt_n=n \\ t_1+t_2+\cdots+t_n=k}} \binom{k}{t_1, t_2, \ldots, t_n} = \binom{n-1}{k-1}. \tag{1}$$

In fact, the number of integer compositions of n is 2^{n-1} and the number with exactly k parts is $\binom{n-1}{k-1}$. If we “forget” the order of the parts, we turn a composition into a partition and the number of compositions corresponding to a given partition becomes a matter of arrangements whose answer is a multinomial coefficient. So $\binom{n-1}{k-1}$ can be expressed as a sum over integer partitions, which is true for any binomial coefficient. We see that the number of terms in the left hand side of (1) is equal to $p(n,k)$.

If k is a divisor of n, then there is only one partition $t_1 + 2t_2 + \cdots + nt_n = n$ such that $t_1 + t_2 + \cdots + t_n = k$ and

$$\binom{k}{t_1, t_2, \ldots, t_n} = 1.$$

On the other hand, if $n \bmod k > 0$ then for any partition $t_1 + 2t_2 + \cdots + nt_n = n$ such that $t_1 + t_2 + \cdots + t_n = k$, we have

$$\binom{k}{t_1, t_2, \ldots, t_n} > 1.$$

Thus by (1), we deduce the following inequality

$$p(n,k) \leqslant \frac{1}{2}\binom{n-1}{k-1} + \frac{1}{2}\delta_{0,n \bmod k}, \tag{2}$$

where $\delta_{i,j}$ is the Kronecker delta.