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On additive complement of a finite set

Sándor Z. Kiss [a,b,*,1], Eszter Rozgonyi [a,2], Csaba Sándor [a,3]

[a] *Institute of Mathematics, Budapest University of Technology and Economics, H-1529 B.O. Box, Hungary*

[b] *Computer and Automation Research Institute of the Hungarian Academy of Sciences, Budapest H-1111, Lágymányosi street 11, Hungary*

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ABSTRACT

We say the sets of nonnegative integers $\mathcal{A}$ and $\mathcal{B}$ are additive complements if their sum contains all sufficiently large integers. In this paper we prove a conjecture of Chen and Fang about additive complement of a finite set by using analytic tools.

* Corresponding author.
E-mail addresses: kisspest@cs.elte.hu (S.Z. Kiss), reszti@math.bme.hu (E. Rozgonyi), csandor@math.bme.hu (C. Sándor).

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1. Introduction

Let $\mathbb{N}$ denote the set of positive integers and let $\mathcal{A} \subseteq \mathbb{N}$ and $\mathcal{B} \subseteq \mathbb{N}$ be finite or infinite sets. Let $R_{\mathcal{A}+\mathcal{B}}(n)$ denote the number of solutions of the equation

$$a + b = n, \quad a \in \mathcal{A},\ b \in \mathcal{B}.$$

We put

$$A(n) = \sum_{\substack{a \leqslant n \\ a \in \mathcal{A}}} 1 \quad \text{and} \quad B(n) = \sum_{\substack{b \leqslant n \\ b \in \mathcal{B}}} 1$$

respectively. We say a set $\mathcal{B} \subseteq \mathbb{N}$ is an additive complement of the set $\mathcal{A} \subseteq \mathbb{N}$ if every sufficiently large $n \in \mathbb{N}$ can be represented in the form $a + b = n$, $a \in \mathcal{A}$, $b \in \mathcal{B}$, i.e., $R_{\mathcal{A}+\mathcal{B}}(n) \geqslant 1$ for $n \geqslant n_0$. Additive complement is an important concept in additive number theory, in the past few decades it was studied by many authors [4,6,8,9]. In [9] Sárközy and Szemerédi proved a conjecture of Danzer [4], namely they proved that for infinite additive complements $\mathcal{A}$ and $\mathcal{B}$ if

$$\limsup_{x \to +\infty} \frac{A(x)B(x)}{x} \leqslant 1,$$

then

$$\liminf_{x \to +\infty} \bigl(A(x)B(x) - x\bigr) = +\infty.$$

In [1] Chen and Fang improved this result and they proved that if

$$\limsup_{x \to +\infty} \frac{A(x)B(x)}{x} > 2, \quad \text{or} \quad \limsup_{x \to +\infty} \frac{A(x)B(x)}{x} < \frac{5}{4},$$

then

$$\lim_{x \to +\infty} \bigl(A(x)B(x) - x\bigr) = +\infty.$$

In the other direction they proved in [2] that for any integer $a \geqslant 2$, there exist two infinite additive complements $\mathcal{A}$ and $\mathcal{B}$ such that

$$\limsup_{x \to +\infty} \frac{A(x)B(x)}{x} = \frac{2a+2}{a+2},$$

but there exist infinitely many positive integers x such that $A(x)B(x) - x = 1$. In [3] they studied the case when $\mathcal{A}$ is a finite set. In this case the situation is different from