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Anisotropy and the integral closure

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ABSTRACT

Let K be a number field and let A be an order in K. The trace map from K to $\mathbf{Q}$ induces a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle : B \times B \to \mathbf{Q}/\mathbf{Z}$ where B is a certain finite abelian group of size $\Delta(A)$. In this article we discuss how one can obtain information about $\mathcal{O}_K$ by purely looking at this symmetric bilinear form. The concepts of *anisotropy* and *quasi-anisotropy*, as defined in another article by the author, turn out to be very useful. We will for example show that under certain assumptions one can obtain $\mathcal{O}_K$ directly from $\langle\ ,\ \rangle$. In this article we will work in a more general setting than we have discussed above. We consider orders over Dedekind domains.

1. Introduction

We will discuss the relation between the new concepts of anisotropy and quasi-anisotropy as defined in [6] and the integral closure of an order in its total quotient ring. These concepts show that in some cases one can find explicit formulas for the integral closure.

All rings in this article are assumed to be commutative.

We will first discuss some practical versions of the theorems which we will prove in this article. First let B be a finite abelian group, with additive notation. Then we define

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the *lower root* of B as

$$\mathrm{lr}(B) = \sum_{r \in \mathbf{Z}} rB \cap B[r],$$

where $B[r] = \{b \in B\colon\ rb = 0\}$. Let α be an algebraic integer and let $K = \mathbf{Q}(\alpha)$ and $A = \mathbf{Z}[\alpha]$. One has a trace map $\mathrm{Tr}_{K/\mathbf{Q}} : K \to \mathbf{Q}$. Now define the *trace dual* of A as

$$A^\dagger = \bigl\{x \in K\colon\ \mathrm{Tr}(xA) \subseteq \mathbf{Z}\bigr\}.$$

It turns out that $A^\dagger$ contains A and that $A^\dagger/A$ is a finite abelian group. Let $\overline{A} = \mathcal{O}_K$ be the integral closure of A in K. Our goal is to determine $\overline{A}$.

The starting point of the theory which we develop in this article, is the following theorem (see Section 9).

Theorem 1.1. *Let $p \in \mathbf{Z}$ be prime and assume that $p > [K : \mathbf{Q}]$. Then we have $p \mid \exp(\overline{A}/A)$ if and only if $p^2 \mid \exp(A^\dagger/A)$, where* exp *stands for the exponent of the corresponding group.*

Using this theorem one can prove that, under the assumption that a certain form is anisotropic or quasi-anisotropic, $\mathcal{O}_K$ corresponds to the lower root of $A^\dagger/A$. For example, one has the following theorem, which uses the concept of anisotropy (see Section 9).

Theorem 1.2. *Let $B = A^\dagger/A$ and suppose that $2 \nmid \#B$. Suppose that $B \cong \mathbf{Z}/m\mathbf{Z} \times \mathbf{Z}/m'\mathbf{Z}$, where $m = \prod_{p\ prime} p^{n(p)}$ and similarly $m' = \prod_{p\ prime} p^{n'(p)}$ such that for all primes p we have $n(p)n'(p) = 0$ or $n(p) + n'(p)$ is odd. Then $\overline{A}/A = \mathrm{lr}(A^\dagger/A)$.*

For example, let α be a root of $x^4 - 20x^3 - 18x^2 + 5x + 6 \in \mathbf{Z}[x]$ and set $A = \mathbf{Z}[\alpha]$. One easily calculates that $A^\dagger/A \cong \mathbf{Z}/5^2\mathbf{Z} \times \mathbf{Z}/(5^3 \cdot 8431)\mathbf{Z}$. The above theorem then gives

$$\overline{A} = A + \frac{3\alpha^2 + 3}{5}\mathbf{Z} + \frac{3\alpha^3 + 4\alpha^2 + 3\alpha + 4}{5}\mathbf{Z}.$$

Using quasi-anisotropy (and some other techniques) one can strengthen the above result. In order to find the integral closure, it is enough to find the integral closure locally. We have the following theorem (see Section 14).

Theorem 1.3. *Let $\mathfrak{m} \subset A$ be a maximal ideal, let $p\mathbf{Z} = \mathfrak{m} \cap \mathbf{Z}$. Define the numbers $n(i)$ such that $(A^\dagger/A)_{\mathfrak{m}} \cong \bigoplus_{i \geqslant 1} (\mathbf{Z}/p^i\mathbf{Z})^{n(i)}$. Suppose that the following conditions are satisfied.*

i. $p > \sum_{i \geqslant 1} n(i)$;
ii. *There exist $i_1, i_2 \in \mathbf{Z}_{\geqslant 1}$ such that*
   - $i_1 \not\equiv i_2 \bmod 2$;