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The projective envelope of a cuspidal representation of a finite linear group

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ABSTRACT

Let ℓ be a prime and let q be a prime power not divisible by ℓ. Put $G = \mathrm{GL}_n(\mathbb{F}_q)$ and fix an irreducible $\bar{\mathbb{F}}_\ell[G]$-representation, $\bar{\pi}$, such that $\bar{\pi}$ is cuspidal but not supercuspidal. We compute the $\mathrm{W}(\bar{\mathbb{F}}_\ell)[G]$-endomorphism ring of the projective envelope of $\bar{\pi}$ under the assumption that $\ell > n > 1$. Our computations provide evidence for a conjecture of Helm relating the Bernstein center to the deformation theory of Galois representations (see [9]).

Contents

1. Introduction

Let p and ℓ be distinct primes and suppose that F is a p-adic field. One of the most celebrated results of modern number theory is the (ℓ-adic) local Langlands correspondence on F. The local Langlands correspondence is a canonical well-behaved bijection

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$$\mathrm{Irr}_\ell\big(\mathrm{GL}_n(F)\big) \to \mathrm{Rep}_\ell^n(\mathrm{W}_F),$$

where $\mathrm{Rep}_\ell^n(\mathrm{W}_F)$ is the set of continuous n-dimensional Frobenius semi-simple representations of the Weil group, W_F, of F over $\bar{\mathbb{Q}}_\ell$ (a fixed algebraic closure of $\mathbb{Q}_\ell$) and $\mathrm{Irr}_\ell(\mathrm{GL}_n(F))$ is the set of irreducible admissible representations over $\bar{\mathbb{Q}}_\ell$ of the general linear group, $\mathrm{GL}_n(F)$ (see [7,10] and Section 32 of Chapter 7 of [3]).

Certainly, part of the importance of the local Langlands correspondence is that notions on one side often correspond nicely with notions on the other. In particular, the notion of semi-simplification on the Weil side corresponds with the notion of supercuspidal support on the general linear side, a bijection known as the (ℓ-adic) semi-simple Langlands correspondence.

The semi-simple local Langlands correspondence is important partially because, unlike the local Langlands correspondence itself, it translates well to the case of modular representations. Indeed, denoting by $\bar{\mathbb{F}}_\ell$ the residue field of $\bar{\mathbb{Q}}_\ell$, Vignéras has shown the following:

Theorem. *There is a unique bijection between supercuspidal supports of* $\mathrm{GL}_n(F)$*-representations over* $\bar{\mathbb{F}}_\ell$ *and* n*-dimensional semi-simple Weil representations over* $\bar{\mathbb{F}}_\ell$ *that is compatible with the semi-simple local Langlands correspondence and reduction modulo* ℓ*.*

Proof. See [17]. □

For obvious reasons, Vignéras' correspondence is often called the ℓ-modular semi-simple local Langlands correspondence.

One is led to consider whether these correspondences can be understood geometrically. To fix ideas, let $\bar{\pi}$ be an irreducible representation of $\mathrm{GL}_n(F)$ over $\bar{\mathbb{F}}_\ell$ such that $\bar{\pi}$ is cuspidal but not supercuspidal (we will avoid the supercuspidal case as it has already been considered by Dat; see Section B.1 of [4], particularly the proposition in B.1.6). Denote by $\bar{\rho}$ the semi-simple Weil representation over $\bar{\mathbb{F}}_\ell$ corresponding to $\bar{\pi}$ via the ℓ-modular semi-simple local Langlands correspondence. Attached to $\bar{\rho}$ is the framed universal deformation ring, $R_{\bar{\rho}}^{\square}$, which parameterizes lifts of $\bar{\rho}$ together with a choice of basis. One is led to consider whether a corresponding algebraic object can be found on the general linear side of local Langlands.

An important construction to this end is the *Bernstein center*, which for any category $\mathcal{A}$, is the endomorphism ring of the identity functor on $\mathcal{A}$. Classically, the Bernstein center has been important in the study of representations of $\mathrm{GL}_n(F)$. In particular, Bernstein and Deligne were able calculate the center of the category, $\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$, of smooth $\mathbb{C}$-representations of $\mathrm{GL}_n(F)$ (see [1]). Moreover, they give a decomposition of the category into a product of blocks (called Bernstein components) and a description of the center of each block, which they show to be a finitely generated $\mathbb{C}$-algebra. These results can also be translated to the field $\bar{\mathbb{Q}}_\ell$.

In consideration of a geometric interpretation of the local Langlands correspondence, Helm has considered the situation for representations over the Witt vec-