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Mean divisibility of multinomial coefficients

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ABSTRACT

Let $m_1, \dots, m_s$ be positive integers. Consider the sequence defined by multinomial coefficients:

$$a_n = \binom{(m_1 + m_2 + \dots + m_s)n}{m_1 n, m_2 n, \dots, m_s n}.$$

Fix a positive integer $k \geqslant 2$. We show that there exists a positive integer $C(k)$ such that

$$\frac{\prod_{n=1}^{t} a_{kn}}{\prod_{n=1}^{t} a_n} \in \frac{1}{C(k)}\mathbb{Z}$$

for all positive integer t, if and only if $GCD(m_1, \dots, m_s) = 1$.
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1. Mean divisibility

A sequence (a_n) $(n = 1, 2, \dots)$ of non-zero integers is *divisible* if $n \mid m$ implies $a_n \mid a_m$. It is *strongly divisible* if $GCD(a_n, a_m) = |a_{GCD(n,m)}|$. Such divisibility attracts number theorists for a long time and a lot of papers dealt with properties of such sequences

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[17,16,4,3,9,1,11]. Primitive divisors of elliptic divisibility sequences and sequences arose in arithmetic dynamics are recently studied in detail [6,12,19]. In this paper, we introduce a weaker terminology which seems not studied before. We say that (a_n) is *almost mean k-divisible*, if there is a positive integer $C = C(k)$ such that $(\prod_{n=1}^t a_{kn})/(\prod_{i=1}^t a_n) \in \frac{1}{C}\mathbb{Z}$ for any positive integer t. In particular, (a_n) is *mean divisible* if $\prod_{n=1}^t a_n \mid \prod_{i=1}^t a_{kn}$ for any positive integer k and t. Clearly if (a_n) is divisible, then it is mean divisible. By definition, if a sequence is almost mean k-divisible for all k with the constant $C(k) = 1$, then it is mean divisible. We are interested in giving non-trivial examples of (almost) mean divisible sequences. In fact, we show that sequences defined by multinomial coefficients give such examples. Let $m_1, \dots, m_s$ be positive integers. A *multinomial sequence* is defined by

$$a_n = \binom{(m_1 + m_2 + \cdots + m_s)n}{m_1 n, m_2 n, \dots, m_s n} = \frac{((m_1 + m_2 + \cdots + m_s)n)!}{(m_1 n)!(m_2 n)! \dots (m_s n)!}.$$

Theorem 1. *If $GCD(m_1, m_2, \dots, m_s) = 1$, then the multinomial sequence is almost mean k-divisible for all k.*

The proof relies on an interesting integral inequality (Lemma 3) and its approximation by Riemann sums. Here are some illustrations:

Corollary 2.

$$\frac{\prod_{n=1}^t \binom{10n}{4n}}{\prod_{n=1}^t \binom{5n}{2n}} \in \frac{1}{11}\mathbb{Z}, \qquad \frac{\prod_{n=1}^t \binom{9n}{3n}}{\prod_{n=1}^t \binom{3n}{n}} \in \frac{1}{5}\mathbb{Z}, \qquad \frac{\prod_{n=1}^t \binom{28n}{4n,8n,16n}}{\prod_{n=1}^t \binom{7n}{n,2n,4n}} \in \mathbb{Z}$$

for any positive integer t.

Readers will see that Figs. 1, 2 and 3 in Section 6 essentially tell why these are true. The constant $C(k)$ is computed by an algorithm based on the proof of Theorem 1. However it is not so easy to identify the set of t's at which the denominator actually appears. For the first example, there are infinitely many t with denominator 11, but the denominator 5 in the second example appears only when $t = 2$. See Section 6 for details. We can also show

Theorem 3. *If $GCD(m_1, m_2, \dots, m_s) > 1$, then the multinomial sequence is not almost mean k-divisible for all k.*

Thus for a given k, a multinomial sequence is almost mean k-divisible if and only if $GCD(m_1, m_2, \dots, m_s) = 1$ holds. For e.g.,

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