Journal of Number Theory 141 (2014) 1–12

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On a certain family of inverse ternary cyclotomic polynomials

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ARTICLE INFO

Article history:
Received 16 October 2013
Received in revised form 10 January 2014
Accepted 10 January 2014
Available online 5 March 2014
Communicated by David Goss

MSC:
11B83
11C08

Keywords:
Inverse ternary cyclotomic polynomial
Height of a polynomial
Flat polynomial

ABSTRACT

We study a family of inverse ternary cyclotomic polynomials Ψ_{pqr} in which $r \leqslant \varphi(pq)$ is a positive linear combination of p and q. We derive a formula for the height of such polynomial and characterize all flat polynomials in this family.

1. Introduction

Let

$$\Phi_n(x) = \prod_{1\leqslant k\leqslant n,\ (k,n)=1} \left(x - e^{2k\pi i/n}\right) = \sum_m a_n(m)x^m$$

be the nth cyclotomic polynomial. The nth inverse cyclotomic polynomial is defined by the formula

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http://dx.doi.org/10.1016/j.jnt.2014.01.019

$$\Psi_n(x) = \frac{x^n - 1}{\Phi_n(x)} = \sum_m c_n(m) x^m.$$

Like for cyclotomic polynomials, for odd primes $p < q < r$, we say that Ψ_{pq} is binary, Ψ_{pqr} is ternary, etc.

Recall that the height of a given polynomial F is the maximal absolute value of its coefficients. We say that polynomial with integral coefficients is flat, if its height equals 1. Traditionally we denote the height of Φ_n by $A(n)$ and the height of Ψ_n by $C(n)$.

Ternary inverse cyclotomic polynomials were studied by P. Moree [5]. He proved that $C(pqr) \leqslant p - 1$ and for every prime $p \geqslant 3$ there are infinitely many pairs (q, r) of primes for which $C(pqr) = p - 1$. Additionally he came up with the following bound (see [5], Theorem 7):

$$C(pqr) \leqslant \max\{\min\{p', q'\}, \min\{q - p', p - q'\}\} \quad \text{for } \deg \Psi_{pqr} < 2qr,$$

where $p' \in \{1, 2, \dots, q - 1\}$ is the inverse of p modulo q and $q' \in \{1, 2, \dots, p - 1\}$ is the inverse of q modulo p. He also found some flat inverse ternary cyclotomic polynomials.

Let us remark that the case $r > \varphi(pq) = \deg \Phi_{pq}$ is trivial, because by the identity $\Psi_{pqr}(x) = \Psi_{pq}(x^r)\Phi_{pq}(x)$ we have $c_{pqr}(ar+b) = a_{pq}(b)c_{pq}(a)$ for $a \geqslant 0$ and $0 \leqslant b < r$. The coefficients of polynomials Φ_{pq} and Ψ_{pq} are well known, so we can evaluate $c_{pqr}(ar + b)$ easily.

Although there is a substantial research on flat ternary cyclotomic polynomials [1–3], we do not know much about flat ternary inverse cyclotomic polynomials. Particularly, no infinite family of such polynomials in which $r \leqslant \varphi(pq)$ was known so far.

In this paper we investigate polynomials Ψ_{pqr} in which $r = \alpha p + \beta q \leqslant \varphi(pq)$, where α and β are positive integers. For this specific type of polynomials we improve some of the results of P. Moree mentioned above. Our main result is the following theorem.

Theorem 1. *Let $r = \alpha p + \beta q \leqslant \varphi(pq)$, where $\alpha, \beta > 0$. Let also $p' \in \{1, 2, \dots, q - 1\}$ be the inverse of p modulo q and $q' \in \{1, 2, \dots, p - 1\}$ be the inverse of q modulo p. Then*

$$C(pqr) = \max\left\{\min\left\{\left\lceil \frac{p'}{\alpha} \right\rceil, \left\lceil \frac{q'}{\beta} \right\rceil\right\}, \min\left\{\left\lceil \frac{q - p'}{\alpha} \right\rceil, \left\lceil \frac{p - q'}{\beta} \right\rceil\right\}\right\}.$$

The above formula is similar to the already mentioned one obtained by P. Moree. However, our theorem does not require the assumption $\deg \Psi_{pqr} < 2qr$. We use Theorem 1 to characterize all flat inverse ternary cyclotomic polynomials Ψ_{pqr} in which r is a positive linear combination of p and q.

Theorem 2. *Let $r = \alpha p + \beta q \leqslant \varphi(pq)$, where $\alpha, \beta > 0$. Then Ψ_{pqr} is flat if and only if at least one of the following conditions holds:*

(a) $\alpha \geqslant \max\{p', q - p', \}$,
(b) $\beta \geqslant \max\{q', p - q', \}$,