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On additive complements. III

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ABSTRACT

Text. Two infinite sequences A and B of non-negative integers are called *additive complements*, if their sum contains all sufficiently large integers. Let $A(x)$ and $B(x)$ be the counting functions of A and B. In 1994, Sárközy and Szemerédi proved that, for additive complements A and B, if $\limsup A(x)B(x)/x \leqslant 1$, then $A(x)B(x) - x \to +\infty$ as $x \to +\infty$. In 2010, the authors generalized this result and proved that if $\limsup A(x)B(x)/x < 5/4$ or $\limsup A(x)B(x)/x > 2$, then $A(x)B(x) - x \to +\infty$ as $x \to +\infty$. In 2011, the authors pointed out that the constant 2 cannot be improved. In this paper, we improve $5/4$ to $3 - \sqrt{3}$.

Video. For a video summary of this paper, please click here or visit http://youtu.be/fSTUWhPJdtw.

1. Introduction

Two infinite sequences A and B of non-negative integers are called *additive complements*, if their sum contains all sufficiently large integers. Let $A(x)$ and $B(x)$ be the counting functions of A and B. For the construction of additive complements A and B with $A(x)B(x) \sim x$, one may refer to [13].

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Starting out from a problem of Hanani and Erdős [4,5], in 1964, Danzer [3] conjectured that, for additive complements A and B, if

$$\limsup_{x\to\infty} \frac{A(x)B(x)}{x} \leqslant 1,$$

then

$$A(x)B(x) - x \to +\infty \quad \text{as } x \to +\infty. \tag{1.1}$$

(See also [6], [9, p. 75] and [11].) In [14], Sárközy and Szemerédi proved this conjecture. In 2010, the authors [7] generalized this result and proved the following result.

Theorem A. *For additive complements A, B, if*

$$\limsup_{x\to\infty} \frac{A(x)B(x)}{x} > 2 \quad \text{or} \quad \limsup_{x\to\infty} \frac{A(x)B(x)}{x} < \frac{5}{4},$$

then (1.1) must hold.

In 2011, the authors pointed out that the constant 2 in above Theorem A cannot be improved. In fact, the following result is proved in [1].

Theorem B. *For any integer a with $a \geqslant 2$, there exist additive complements A, B such that*

$$\limsup_{x\to\infty} \frac{A(x)B(x)}{x} = \frac{2a+2}{a+2},$$

but there exist infinitely positive integers x such that $A(x)B(x) - x = 1$.

Let C_a be the supremum of real numbers δ such that, for any additive complements A, B, if

$$\limsup_{x\to\infty} \frac{A(x)B(x)}{x} < \delta,$$

then (1.1) must hold. By Theorems A and B, we have $5/4 \leqslant C_a \leqslant 3/2$ (taking $a = 2$ in Theorem B).

In this paper, we improve the lower bound of C_a from $5/4$ to $3 - \sqrt{3}$. That is, the following result is proved.

Theorem 1.1. *For additive complements A, B, if*

$$\limsup_{x\to\infty} \frac{A(x)B(x)}{x} < 3 - \sqrt{3},$$

then $A(x)B(x) - x \to +\infty$ as $x \to +\infty$.