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New congruences modulo powers of 2 for broken 3-diamond partitions and 7-core partitions

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A R T I C L E I N F O

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A B S T R A C T

Let $\Delta_k(n)$ denote the number of broken k-diamond partitions of n for a fixed positive integer k. Recently, Radu and Sellers conjectured that for all $\alpha \geqslant 1$ and $n \geqslant 0$, $\Delta_3(\lambda_\alpha)\Delta_3(2^{\alpha+2}n + \lambda_{\alpha+2}) \equiv \Delta_3(\lambda_{\alpha+2})\Delta_3(2^\alpha n + \lambda_\alpha) \pmod{2^\alpha}$, where $\lambda_\alpha = \frac{2^{\alpha+1}+1}{3}$ if α is even and $\lambda_\alpha = \frac{2^\alpha+1}{3}$ if α is odd. Radu and Sellers proved that this conjecture is true for $\alpha = 1$. In this work, we show that this conjecture holds for $\alpha = 2$. We also prove that $\Delta_3(\lambda_\alpha) \equiv (-1)^{[\frac{\alpha}{2}]} \pmod 4$ which yields $\Delta_3(\lambda_\alpha) \equiv 1 \pmod 2$. This congruence was conjectured by Radu and Sellers. Furthermore, we also deduce some new Ramanujan-type congruences modulo 2 and 4 for 7-core partitions.

1. Introduction

The aim of this paper is to derive some congruences modulo powers of 2 for the number of broken 3-diamond partitions and the number of 7-cores. We give a proof of part of a conjecture given by Radu and Sellers. We also generalize some parity results for the number of broken 3-diamond partitions proved by Radu and Sellers and establish some new congruences for the number of 7-cores.

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Let us begin with some notation and terminology on q-series and partitions. Throughout this paper, let $|q|<1$. We use the standard notation

$$(a;q)_\infty=\prod_{k=0}^{\infty}\left(1-aq^k\right) \tag{1.1}$$

and often write

$$(a_1,a_2,\dots,a_n;q)_\infty=(a_1;q)_\infty(a_2;q)_\infty\cdots(a_n;q)_\infty. \tag{1.2}$$

Recall that the Ramanujan theta function $f(a,b)$ is defined by

$$f(a,b)=\sum_{n=-\infty}^{\infty}a^{n(n+1)/2}b^{n(n-1)/2}, \tag{1.3}$$

where $|ab|<1$. The Jacobi triple product identity can be restated as

$$f(a,b)=(-a,-b,ab;ab)_\infty. \tag{1.4}$$

Three special cases of (1.3) are defined by

$$\phi(q)=f(q,q)=\sum_{n=-\infty}^{\infty}q^{n^2}, \tag{1.5}$$

$$\psi(q)=f\left(q,q^3\right)=\sum_{n=0}^{\infty}q^{\frac{n(n+1)}{2}} \tag{1.6}$$

and

$$f(-q)=f\left(-q,-q^2\right)=\sum_{n=-\infty}^{\infty}(-1)^nq^{n(3n-1)/2}=(q;q)_\infty. \tag{1.7}$$

In this paper, for any positive integer n, we use f_n to denote $f(-q^n)$, that is,

$$f_n=\left(q^n;q^n\right)_\infty=\prod_{k=1}^{\infty}\left(1-q^{nk}\right). \tag{1.8}$$

By (1.3), (1.4), (1.5) and (1.6), we have

$$\phi(q)=\frac{f_2^5}{f_1^2f_4^2},\qquad \psi(q)=\frac{f_2^2}{f_1}. \tag{1.9}$$

A combinatorial study guided by MacMahon's Partition Analysis led Andrews and Paule [1] to the construction of a new class of directed graphs called broken k-diamond