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Generalizing Zeckendorf's Theorem to f-decompositions ☆

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Text. A beautiful theorem of Zeckendorf states that every positive integer can be uniquely decomposed as a sum of non-consecutive Fibonacci numbers $\{F_n\}$, where $F_1 = 1$, $F_2 = 2$ and $F_{n+1} = F_n + F_{n-1}$. For general recurrences $\{G_n\}$ with nonnegative coefficients, there is a notion of a legal decomposition which again leads to a unique representation. We consider the converse question: given a notion of legal decomposition, construct a sequence $\{a_n\}$ such that every positive integer can be uniquely decomposed as a sum of a_n's. We prove this is possible for a notion of legal decomposition called f-decompositions. This notion generalizes existing notions such as base-b representations, Zeckendorf

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Recurrence relations
Stirling numbers of the first kind
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decompositions, and the factorial number system. Using this new perspective, we expand the range of Zeckendorf-type results, generalizing the scope of previous research. Finally, for specific classes of notions of decomposition we prove a Gaussianity result concerning the distribution of the number of summands in the decomposition of a randomly chosen integer.

Video. For a video summary of this paper, please click here or visit http://youtu.be/hnYJwvOfzLo.

Contents

1. Introduction

The Fibonacci numbers are a very well known sequence, whose properties have fascinated mathematicians for centuries. Zeckendorf [Ze] proved an elegant theorem stating that every positive integer can be written uniquely as the sum of non-consecutive Fibonacci numbers $\{F_n\}$, where $F_0 = 1$, $F_1 = 2$ are the first two terms[1] and $F_n = F_{n-1} + F_{n-2}$. More is true, as the number of summands for integers in $[F_n, F_{n+1})$ converges to a normal distribution as $n \to \infty$. These results have been generalized to positive linear recurrence sequences of the form

$$G_{n+1} = c_1 G_n + \cdots + c_L G_{n+1-L}, \tag{1.1}$$

where $L, c_1, \ldots, c_L$ are nonnegative and L, c_1 and c_L are positive. For every such recurrence relation there is a notion of "legal decomposition" with which all positive integers have a unique decomposition as a nonnegative integer linear combination of terms from the sequence, and the distribution of the number of summands of integers

[1] We don't start the Fibonacci numbers $F_0 = 0$, $F_1 = 1$, $F_2 = 1$ because doing so would lead to multiple decompositions for some positive integers.