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A characterization of Jacobi cusp forms of certain types

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ABSTRACT

In this paper, we will give a sufficient condition for $f \in J_{k,m}(\Gamma_0(N))$ (m, N: both squarefree with $(2m, N) = 1$) to be cuspidal by using Jacobi Eisenstein series and their explicit Fourier coefficients.

1. Introduction

The authors of [4,9] showed that for a modular form f of even integral weight $k \geqslant 2$ on a congruence subgroup $\Gamma_0(N)$ with $N \geqslant 1$, if the Fourier coefficients $a(n)$ ($n \geqslant 1$) of f satisfy $a(n) \ll_{f,\epsilon} n^{(k-1)/2+\epsilon}$ ($\epsilon > 0$), then f must be a cusp form.

In [6] similar results for the degree two Siegel modular forms on the full Siegel modular group and for the Jacobi forms on the full Jacobi group were given also in terms of the growth of their Fourier coefficients. The authors of [6] mainly used the structure of the Jacobi Eisenstein space and the growth of the Fourier coefficients of the Jacobi Eisenstein series which were given in [2].

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The purpose of this paper is to generalize the result on Jacobi forms from [6]. More precisely, a sufficient condition for $f \in J_{k,m}(\Gamma_0(N))$ (m, N: both squarefree with $(2m, N) = 1$) to be a cusp form will be given. In order to accomplish this we will investigate the structure of the Jacobi Eisenstein space and compute the Fourier coefficients of the Jacobi Eisenstein series. Using these we will give a proof of the main theorem.

When $m = 1$, the result here recovers the result for $k \geqslant 4$ in [5] which was done not using the structure of the Jacobi Eisenstein space but using an isomorphism between $J_{k,1}(\Gamma_0(N))$ and $M^+_{k-1/2}(\Gamma_0(4N))$ for even k and odd N stated in Theorem 8 of [7]. Moreover, if we use that isomorphism of [7] in the main result obtained here, we can regain the corresponding result about $M^+_{k-1/2}(\Gamma_0(4N))$ for even $k \geqslant 4$ and odd N which was proven in [1].

2. Main result

Recall the Jacobi group $G^J = \mathrm{SL}_2(\mathbb{R}) \ltimes (\mathbb{R}^2 \cdot \mathrm{S}^1)$. For fixed integers k and m, we define a slash operator $|_{k,m}$ of G^J on the space of functions $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ given by

$$\begin{aligned}&(\phi|_{k,m}\xi)(\tau, z)\\&\quad = s^m (c\tau + d)^{-k} e^m\left(\frac{-c(z + \lambda\tau + \mu)^2}{c\tau + d} + \lambda^2\tau + 2\lambda z + \lambda\mu\right)\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right),\end{aligned}$$

where $e^m(x) = e^{2\pi imx}$ and $\xi = (\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), (\lambda\ \mu), s) \in G^J$.

Definition 2.1. A holomorphic function $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ is called a Jacobi form of weight k and index m ($k, m \in \mathbb{N}$) on $\Gamma_0(N)$ provided ϕ satisfies the following conditions:

(i) $\phi|_{k,m}\xi = \phi$ for all $\xi = (\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), (\lambda\ \mu)) \in \Gamma_0(N) \ltimes \mathbb{Z}^2$,
(ii) for every $M \in \Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$, $\phi|_{k,m} M$ has a Fourier expansion of the form

$$(\phi|_{k,m}M)(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ 4nm \geqslant r^2 n_M}} c_M(n, r) q^{n/n_M} \zeta^r \quad \left(q = e^{2\pi i\tau},\ \zeta = e^{2\pi iz}\right),$$

where n_M is the least positive integer such that $M\left(\begin{smallmatrix} 1 & n_M \\ 0 & 1 \end{smallmatrix}\right)M^{-1} \in \Gamma_0(N)$.

We define the constant part $\sum c_M(n, r) q^{n/n_M} \zeta^r$ where the summation is taken over all $n, r \in \mathbb{Z}$ with $4nm = r^2 n_M$. If the constant part of $\phi|_{k,m}M$ vanishes for every $M \in \Gamma(1)$, ϕ is called a Jacobi cusp form of weight k and index m on $\Gamma_0(N)$. The vector space of all Jacobi forms of weight k and index m on $\Gamma_0(N)$ is denoted as $J_{k,m}(\Gamma_0(N))$. And the subspace of all Jacobi cusp forms of weight k and index m on $\Gamma_0(N)$ is denoted by $J^{\mathrm{cusp}}_{k,m}(\Gamma_0(N))$.