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Spherical designs and heights of Euclidean lattices

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ABSTRACT

We study the connection between the theory of spherical designs and the question of extrema of the height function of lattices. More precisely, we show that a full-rank n-dimensional Euclidean lattice Λ, all layers of which hold a spherical 2-design, realises a stationary point for the height $h(\Lambda)$, which is defined as the first derivative at the point 0 of the spectral zeta function of the associated flat torus $\zeta(\mathbf{R}^n/\Lambda)$. Moreover, in order to find out the lattices for which this 2-design property holds, a strategy is described which makes use of theta functions with spherical coefficients, viewed as elements of some space of modular forms. Explicit computations in dimension $n \leqslant 7$, performed with Pari/GP and Magma, are reported.

1. Introduction

The aim of this article is to investigate (local) extremality properties of the height on the set of lattices of covolume 1, and to describe its stationary points in terms of spherical designs.

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The *height* of a Euclidean lattice Λ is defined as the derivative at the point $s = 0$ of the spectral zeta function of the flat torus associated to Λ. More generally, if (X, g) is a compact connected Riemannian manifold without boundary, the spectrum of the associated Laplace operator is a discrete sequence of real numbers $0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$, converging to infinity, and the *spectral zeta function* is then defined, for $\mathrm{Re}(s) \gg 0$, by the series

$$\zeta_{(X,g)}(s) = \sum_{i=1}^{\infty} \frac{1}{\lambda_i^s}. \tag{1}$$

This expression is well-known to admit a meromorphic continuation to $\mathbf{C}$, holomorphic at 0 (see [5,14]). This allows to give meaning to the determinant of the Laplacian, *via* the standard zeta regularisation process

$$\det \Delta_g := \exp\bigl(-\zeta'_{(X,g)}(0)\bigr), \tag{2}$$

which indeed can be interpreted as the formal infinite product of the λ_i's. Then the height of (X, g) is simply $h(X) = \zeta'_{(X,g)}(0)$. A natural problem arises at this point, namely to maximise $\det \Delta_g$ on a given manifold X as the metric g varies (with fixed volume) and characterise the optimal metrics (see [21]).

When X is a flat torus $\mathbf{R}^n/\Lambda$ associated to a full-rank lattice Λ in the Euclidean space $\mathbf{R}^n$, then this question seems to be more tractable, though far from trivial. Indeed, for such a lattice Λ one can construct the *Epstein zeta function*

$$Z(\Lambda, s) := \sum_{0 \neq x \in \Lambda} \frac{1}{\|x\|^{2s}}, \quad s \in \mathbf{C} \text{ with } \mathrm{Re}(s) > n/2, \tag{3}$$

which is well-known (since Epstein [13]) to admit a meromorphic continuation to the complex plane with a simple pole at $s = n/2$. Besides, the eigenvalues of the Laplacian on $\mathbf{R}^n/\Lambda$ are exactly $4\pi^2\|x\|^2$, for x in the standard dual

$$\Lambda^* = \bigl\{y \in \mathbf{R}^n \bigm| (x \cdot y) \in \mathbf{Z} \text{ for all } x \in \Lambda\bigr\} \tag{4}$$

(see [7] and [24]), whence we have the following identity between the two zeta functions

$$\zeta_{\mathbf{R}^n/\Lambda}(s) = (2\pi)^{-2s} Z\bigl(\Lambda^*, s\bigr), \tag{5}$$

which gives immediately the one for the height of Λ:

$$h(\Lambda) = \zeta'_{\mathbf{R}^n/\Lambda}(0) = Z'\bigl(\Lambda^*, 0\bigr) + 2\log(2\pi). \tag{6}$$

Since for $c > 0$ we have that $Z(c\Lambda, s) = c^{-2s} Z(\Lambda, s)$, the question of the minima of the height makes sense only if we restrict ourselves to lattices of fixed covolume, usually 1. From now on, let $\mathcal{L}_n^\circ$ denote the set of lattices of determinant 1, hence of covolume 1: