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Regular decomposition of ordinarity in generic exponential sums

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ABSTRACT

In [16] (1993) and [18] (2004) Wan establishes a decomposition theory for the generic Newton polygon associated to a family of L-functions of n-dimensional exponential sums over finite fields. In this work we generalize the star, parallel hyperplane and collapsing decomposition, demonstrating that each is a generalization of a complete regular decomposition.

1. Introduction

In [2] Adolphson and Sperber established a combinatorial lower bound for the generic Newton polygon attached to a family of L-functions of n-dimensional exponential sums over finite fields. Notably this lower bound is independent of the character of the finite field. In the case of toric hypersurfaces, the bound required the use of Hodge numbers. For this reason, the bound was called the Hodge polygon. Generic Newton polygons which coincide with the Hodge polygon are called generically ordinary. In [2] they also conjectured conditions when generic ordinarity holds.

Wan showed that Adolphson and Sperber's conjecture is in general false [16]. Using maximizing functions from linear programming, he obtained several decomposition theorems that, in effect, decompose the property of generic ordinarity [15,16,18].

In this paper we show that the star and parallel hyperplane decomposition appearing in [16] and the collapsing decomposition that appears in [18] are each instances of a more general decomposition

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type referred to in linear programming as a regular decomposition. This application of regular decompositions was first conjectured, by Wan in the case of toric hypersurfaces [19]. One must note the facial decomposition and boundary decomposition in [16] are not special cases of the regular decomposition. In fact, the defining property of a regular decomposition is that it mimics a facial decomposition. Both the facial and boundary decompositions are heavily used in showing that regular decomposition decomposes the property of ordinarity.

Throughout this work several examples of ordinarity, and several decompositions are provided. We conclude with a demonstration of the regular decomposition in the case of Deligne polytopes.

1.1. Definition of L-function

Let p be a prime and $q = p^a$ for some positive integer a. Let $\mathbb{F}_q$ be the finite field of q elements. For each positive integer k, let $\mathbb{F}_{q^k}$ be the finite extension of $\mathbb{F}_q$ of degree k. Let ζ_p be a fixed primitive p-th root of unity in the complex numbers. For any Laurent polynomial $f(x_1, \dots, x_n) \in \mathbb{F}_q[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, we form the exponential sum

$$S_k^*(f) = \sum_{x_i \in \mathbb{F}_{q^k}^*} \zeta_p^{\mathrm{Tr}_k f(x_1, \dots, x_n)}, \tag{1}$$

where $\mathbb{F}_{q^k}^*$ denotes the set of non-zero elements in $\mathbb{F}_{q^k}$ and Tr_k denotes the trace map from $\mathbb{F}_{q^k}$ to the prime field $\mathbb{F}_p$.

By a theorem of Dwork–Bombieri–Grothendieck, the following generating L-function is a rational function [6,9]:

$$L^*(f, T) = \exp\left(\sum_{k=1}^{\infty} S_k^*(f) \frac{T^k}{k}\right). \tag{2}$$

We may write f as:

$$f = \sum_{j=1}^{J} a_j x^{V_j}, \quad a_j \neq 0,$$

where each $V_j = (v_{1j}, \dots, v_{nj})$ is a lattice point in $\mathbb{Z}^n$ and the power x^{V_j} is the product $x_1^{v_{1j}} \cdots x_n^{v_{nj}}$. Let $\Delta(f)$ be the convex closure in $\mathbb{R}^n$ generated by the origin and the lattice points V_j $(1 \leqslant j \leqslant J)$. This is called the Newton polyhedron of f. Without loss of generality we may always assume that $\Delta(f)$ is n-dimensional.

For δ a subset of $\{V_1, \dots, V_J\}$, we define the restriction of f to δ to be the Laurent polynomial

$$f_\delta = \sum_{V_j \in \delta} a_j x^{V_j}.$$

For our purposes, we will generally take δ to be a sub-polytope or a face of Δ. This polytope structure suggests the following definition:

Definition 1.1. The Laurent polynomial f is called non-degenerate or Δ-regular if for each closed face δ of $\Delta(f)$ of arbitrary dimension which does not contain the origin, the n partial derivatives

$$\left\{\frac{\partial f_\delta}{\partial x_1}, \dots, \frac{\partial f_\delta}{\partial x_n}\right\}$$

have no common zeros with $x_1 \cdots x_n \neq 0$ over the algebraic closure of $\mathbb{F}_q$.