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Asymptotic expansions related to Glaisher–Kinkelin constant based on the Bell polynomials

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ABSTRACT

By using the Bell polynomials, we present a class of asymptotic expansions related to Glaisher–Kinkelin constant.

Throughout the present investigation, $\mathbb{N}$ denotes the set of all positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The Glaisher–Kinkelin constant $A = 1.28242712\ldots$ is defined by

$$\lim_{n\to\infty} n^{-n^2/2-n/2-1/12} e^{n^2/4} \prod_{k=1}^{n} k^k = A \tag{1}$$

(see [8,9,14]), as well as

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$$\lim_{n\to\infty}\frac{G(n+1)}{n^{n^2/2-1/12}(2\pi)^{n/2}e^{-3n^2/4}}=\frac{e^{1/12}}{A}, \tag{2}$$

where $G(n)$ is the Barnes G-function [1].

The Glaisher–Kinkelin constant A has closed-form representations

$$\begin{aligned} A &= e^{\frac{1}{12}-\zeta'(-1)} \\ &= (2\pi)^{1/12}\big[e^{\gamma\pi^2/6-\zeta'(2)}\big]^{1/(2\pi^2)} \end{aligned}$$

(see [4, p. 129, Eq. (3.22)]), where $\zeta'(z)$ is the derivative of the Riemann zeta function $\zeta(z)$ (see [5]).

The Glaisher–Kinkelin constant A appears in a number of sums and integrals, especially those involving gamma functions and zeta functions. Finch introduced this constant A in a section of his book [7, pp. 135–138].

Define the sequence $(A_n)_{n\in\mathbb{N}}$ by

$$A_n := n^{-n^2/2-n/2-1/12}e^{n^2/4}\prod_{k=1}^{n}k^k. \tag{3}$$

Very recently, Chen [3, Theorem 1] established the asymptotic expansion of the sequence $(\ln A_n)_{n\in\mathbb{N}}$:

$$\begin{aligned} \ln A_n &= \sum_{k=1}^{n}k\ln k-\left(\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}\right)\ln n+\frac{n^2}{4} \\ &\sim \ln A-\sum_{k=1}^{\infty}\frac{B_{2k+2}}{2k(2k+1)(2k+2)}\frac{1}{n^{2k}}, \end{aligned} \tag{4}$$

where B_k ($k\in\mathbb{N}_0$) are the n-th Bernoulli numbers defined by the following generating function (see, for example, [12, Section 1.6] and [13, Section 1.7]):

$$\frac{z}{e^z-1}=\sum_{k=0}^{\infty}B_k\frac{z^k}{k!}\quad\big(|z|<2\pi\big). \tag{5}$$

The asymptotic representation (4) can be rewritten as

$$1^12^2\cdots n^n\sim A\cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}}e^{-\frac{n^2}{4}}\exp\left(\sum_{j=1}^{\infty}\frac{a_j}{n^j}\right), \tag{6}$$

where

$$a_j=-\frac{B_{j+2}}{j(j+1)(j+2)}\quad(j\in\mathbb{N}). \tag{7}$$

Namely,

$$\begin{aligned} 1^12^2\cdots n^n\sim A\cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}}e^{-\frac{n^2}{4}}\exp\bigg(&\frac{1}{720n^2}-\frac{1}{5040n^4}+\frac{1}{10\,080n^6} \\ &-\frac{1}{9504n^8}+\frac{691}{3\,603\,600n^{10}}-\frac{1}{1872n^{12}}+\cdots\bigg). \end{aligned} \tag{8}$$