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Bounding the degree of Belyi polynomials

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ABSTRACT

Text. Belyi's theorem states that a Riemann surface X, as an algebraic curve, is defined over $\overline{\mathbb{Q}}$ if and only if there exists a holomorphic function B taking X to $P^1\mathbb{C}$ with at most three critical values $\{0, 1, \infty\}$. By restricting to the case where $X = P^1\mathbb{C}$ and our holomorphic functions are Belyi polynomials, for an algebraic number λ, we define a Belyi height $\mathcal{H}(\lambda)$ to be the minimal degree of the set of Belyi polynomials with $B(\lambda) \in \{0, 1\}$. We prove for non-zero λ with non-zero p-adic valuation, the Belyi height of λ is greater than or equal to p using the combinatorics of Newton polygons. We also give examples of algebraic numbers with relatively low height and show that our bounds are sharp.

Video. For a video summary of this paper, please click here or visit http://www.youtube.com/watch?v=MJAodACJ4kM.

1. Introduction

In this paper, we fix an algebraic closure of p-adic numbers and denote it as $\overline{\mathbb{Q}}_p$. We denote an embedded algebraic closure of the rational numbers in $\overline{\mathbb{Q}}_p$ as $\overline{\mathbb{Q}}$. A polynomial $B(x)$ in $\overline{\mathbb{Q}}[x]$ is said to have a critical point at x_i if its derivative $B'(x)$ vanishes at x_i. We say $B(x)$ has a critical value of $B(x_i)$ when x_i is a critical point. A polynomial is said to be a *general Belyi polynomial* if its critical values are contained in $\{0, 1\}$. Since composing a general Belyi polynomial with any linear factor $(\gamma x - \alpha)$ yields another general Belyi polynomial, we normalize our set of polynomials by requiring $B(0), B(1) \in \{0, 1\}$.

Definition 1. A polynomial $B(x) \in \overline{\mathbb{Q}}[x]$ is said to be a *normalized Belyi polynomial* or *Belyi polynomial* if $B(0), B(1) \in \{0, 1\}$ and $\{B(x_i)\colon\ B'(x_i) = 0\} \subset \{0, 1\}$.

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Equivalently, we note that $B(x)$ is a Belyi polynomial if

(i) $B(0), B(1) \in \{0, 1\}$, and
(ii) $B'(x) \mid B(x)(1 - B(x))$.

We call these the two Belyi conditions. With these conditions, a Belyi polynomial composed with a linear factor $(\gamma x - \alpha)$ is a Belyi polynomial if and only if $B(\gamma), B(\gamma - \alpha) \in \{0, 1\}$. For a fixed Belyi polynomial there exist finitely many linear factors we may compose with to yield a Belyi polynomial. This finiteness condition is essential to define a Belyi height with the property that there exist finitely many Belyi polynomials of a given height.

Example 1. The simplest examples of Belyi polynomials are $f(x) = x^n$, $f(x) = 1 - x$, and

$$B_{a,b}(x) = b^b a^{-a}(b - a)^{-(b-a)} x^a (1 - x)^{b-a},$$

with a and $(b - a)$ as positive integers.

The Belyi polynomial $B_{a,b}(x)$ maps $\{\frac{a}{b}, 0, 1\}$ to $\{0, 1\}$. When we compose $B_{a,b}(x)$ with certain polynomials $C(x)$ the result, $B_{a,b}(C(x))$, has fewer critical values than $C(x)$. Specifically, when $C(x)$ satisfies the first Belyi condition and has a critical value of $\frac{a}{b}$, composing with $B_{a,b}$ reduces the number of critical values.

Example 2. The Chebyshev polynomials of the first kind, defined as

$$T_0(x) = 1, \qquad T_1(x) = x, \qquad T_{n+1}(x) = 2x \cdot T_n(x) - T_{n-1}(x)$$

for $n \geqslant 1$ have critical values contained in $\{-1, 1\}$ and $T_n(1), T_n(-1) \in \{-1, 1\}$. Therefore $\frac{1}{2}(T_n(x) + 1)$ are general Belyi polynomials and $\frac{1}{2}(T_n(2x - 1) + 1)$ are Belyi polynomials.

This example is studied in detail in [1] where the normalization of Belyi polynomials is done with respect to $\{-1, 1\}$ instead of $\{0, 1\}$.

Example 3. The composition of any two Belyi polynomials is a Belyi polynomial.

This example is a simple application of the chain rule and gives the set of Belyi polynomials a monoid structure under composition with identity of x. This structure has been used to study absolute Galois groups in number theory [12,4] and dynamical systems [9].

Belyi polynomials belong to the larger set of Belyi functions. A Belyi function f maps a Riemann surface X to the Riemann sphere $P^1\mathbb{C}$ with critical values contained in $\{0, 1, \infty\}$. Grothendieck was drawn into this subject because of Belyi's theorem [3], which states a Riemann surface X is defined over $\overline{\mathbb{Q}}$ if and only if there exists a Belyi function mapping X to $P^1\mathbb{C}$. This marked the beginning of his program on dessins d'enfants [10], which is directly related to Belyi functions due to the well-known categorical equivalence between the two.

In the case when $X = P^1\mathbb{C}$ we normalize Belyi functions by requiring the set $\{0, 1, \infty\}$ be mapped to $\{0, 1, \infty\}$. As a corollary [1] of the Riemann Existence Theorem [8] there exist finitely many normalized Belyi functions that map $P^1\mathbb{C}$ to $P^1\mathbb{C}$ of degree at most n, where *degree* is the cardinality of the pre-image of a point in $P^1\mathbb{C} \setminus \{0, 1, \infty\}$. This means there are finitely many normalized Belyi polynomials of a given degree, hence finitely many algebraic numbers mapped to zero or one by normalized Belyi polynomials of degree d. The question we address in this paper is the following: for fixed $\lambda \in \overline{\mathbb{Q}}$, what is the minimal degree of normalized Belyi polynomials that map λ to zero or one? We call this minimum the *Belyi height* of a number and denote it as $\mathcal{H}(\lambda)$. In [7], an upper bound of $\mathcal{H}(\lambda)$ is given, in addition to bounds for the case when X is an elliptic curve. In this paper, we provide