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Connections between $p = x^2 + 3y^2$ and Franel numbers ☆

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ABSTRACT

The Franel numbers are given by $f_n = \sum_{k=0}^{n} \binom{n}{k}^3$ $(n = 0, 1, 2, \ldots)$. Let $p > 3$ be a prime. When $p \equiv 1 \pmod 3$ and $p = x^2 + 3y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 1 \pmod 3$, we show that

$$\sum_{k=0}^{p-1} \frac{f_k}{2^k} \equiv \sum_{k=0}^{p-1} \frac{f_k}{(-4)^k} \equiv 2x - \frac{p}{2x} \pmod{p^2}.$$

We also prove that if $p \equiv 2 \pmod 3$ then

$$\sum_{k=0}^{p-1} \frac{f_k}{2^k} \equiv -2 \sum_{k=0}^{p-1} \frac{f_k}{(-4)^k} \equiv \frac{3p}{\binom{(p+1)/2}{(p+1)/6}} \pmod{p^2}.$$

In addition, we propose several related conjectures for further research.

1. Introduction

Let $p \equiv 1 \pmod 4$ be a prime and write $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 2$. A famous result of Gauss (cf. B.C. Berndt, R.J. Evans and K.S. Williams [BEW, (9.0.1)]) states

$$\binom{(p-1)/2}{(p-1)/4} \equiv 2x \pmod p,$$

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which was refined by S. Chowla, B. Dwork and R.J. Evans [CDE] as follows:

$$\binom{(p-1)/2}{(p-1)/4} \equiv \frac{2^{p-1}+1}{2}\left(2x-\frac{p}{2x}\right) \pmod{p^2}.$$

In 2010 J.B. Cosgrave and K. Dilcher [CD] even determined $\binom{(p-1)/2}{(p-1)/4}$ mod p^3. The author [Su11a, Conjecture 5.5] conjectured that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{8^k} \equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k} \equiv \left(\frac{2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{32^k} \equiv \left(\frac{2}{p}\right)\left(2x-\frac{p}{2x}\right) \pmod{p^2}$$

(where $(\frac{\cdot}{p})$ denotes the Legendre symbol), and this was confirmed by the author's twin brother Z.-H. Sun [S] with the help of Legendre polynomials. Furthermore, the author [Su12] proved that

$$\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^2}{8^k} \equiv 2\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^2}{(-16)^k} \equiv \left(\frac{2}{p}\right)\left(\frac{p}{2x}-x\right) \pmod{p^2}.$$

When $p \equiv 3 \pmod 4$ is a prime, the author [Su13b] showed that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{8^k} \equiv -\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k} \equiv \frac{(-1)^{(p+1)/4}2p}{\binom{(p+1)/2}{(p+1)/4}} \pmod{p^2}.$$

For $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, we have the combinatorial identities

$$\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n} \quad \text{and} \quad \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3 = (-1)^n\binom{2n}{n}\binom{3n}{n}$$

(see, e.g., [G, (3.66) and (6.6)]). Note that $\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3 = 0$ for $n = 1, 3, 5, \ldots$. A conjecture of the author [Su11b, Conjecture 5.13] states that if $p > 3$ is a prime then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{24^k} \equiv \left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-216)^k} \equiv \begin{cases} \binom{2(p-1)/3}{(p-1)/3} \pmod{p^2} & \text{if } p \equiv 1 \pmod 3, \\ p/\binom{2(p+1)/3}{(p+1)/3} \pmod{p^2} & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

It is known that for any prime $p \equiv 1 \pmod 3$ we can write $4p = u^2 + 27v^2$ with $u, v \in \mathbb{Z}$ and $u \equiv 1 \pmod 3$, and we have

$$\binom{2(p-1)/3}{(p-1)/3} \equiv \frac{p}{u} - u \pmod{p^2}$$

(cf. [CD, Theorem 6]).

In [Su13a] the author introduced the polynomials $S_n(x) = \sum_{k=0}^{n}\binom{n}{k}^4 x^k$ $(n = 0, 1, 2, \ldots)$ and posed 13 related conjectures one of which states that for any prime $p > 2$ we have

$$\sum_{n=0}^{p-1} S_n(12) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{12} \text{ and } p = x^2 + y^2\ (3 \nmid x), \\ (\frac{xy}{3})4xy \pmod{p^2} & \text{if } p \equiv 5 \pmod{12} \text{ and } p = x^2 + y^2\ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$