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Modularity of abelian varieties over $\mathbf{Q}$ with bad reduction in one prime only

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ABSTRACT

We show that certain abelian varieties over $\mathbf{Q}$ with bad reduction at one prime only are modular by using methods based on the tables of Odlyzko and class field theory.

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1. Introduction

In this article we consider abelian varieties over $\mathbf{Q}$ with bad reduction in one prime only. Fontaine [Fon85] and Abrashkin [Abr87] showed that abelian varieties over $\mathbf{Q}$ (and some other number fields with small discriminant) with good reduction everywhere do not exist. Abelian varieties over $\mathbf{Q}$ with bad reduction in one prime only were then considered in [Sch05] and [BK01]. The following is a summary of results proven in [Sch05, Sch09, Sch11]:

- For $p \in \{2, 3, 5, 7, 13\}$, there do not exist non-zero semi-stable abelian varieties over $\mathbf{Q}$ with good reduction outside p .
- For $p \in \{2, 3, 5\}$, there do not exist non-zero abelian varieties over $\mathbf{Q}$ with good reduction outside p that become semi-stable over an at most tamely ramified extension at p of $\mathbf{Q}$.
- Every semi-stable abelian variety over $\mathbf{Q}$ with good reduction outside 11 is isogenous to a product of copies of $J_0(11)$.
- Every semi-stable abelian variety over $\mathbf{Q}$ with good reduction outside 23 is isogenous to a product of copies of $J_0(23)$.

In any of these results, every such non-zero abelian variety is *modular*: let f be a newform of weight 2 and level N . Then one can associate to f an abelian variety A_f over $\mathbf{Q}$ and hence a representation $\rho_{A_f, \ell} : G_{\mathbf{Q}} \rightarrow \text{Aut}(T_{\ell}(A_f))$. An abelian variety that is isogenous to a product of abelian varieties of the form A_f is called a *(classical) modular abelian variety*.

We want to extend these existence results by proving the modularity of any abelian variety A over $\mathbf{Q}$ with good reduction outside a fixed prime p that satisfies some ramification condition on the Galois representation $\rho_{A, \ell} : G_{\mathbf{Q}} \rightarrow \text{Aut}(T_{\ell}(A))$ at p . The ramification condition on $\rho_{A, \ell}$ will replace the ramification condition of semi-stability at p .

The above results indicate that we have to consider abelian varieties that become semi-stable over an extension that is not necessarily at most tamely ramified at either 2, 3, 5, 7 or 13. Even though such an extension may not be at most tamely ramified, the methods we use to prove modularity of certain abelian varieties over $\mathbf{Q}$ require that we do bound the ramification at p in some way. The next definition provides a method to do this in a fairly canonical way using ramification groups with upper numbering (see Section 2.1):

Definition 1.1. Let $i \in \mathbf{R}_{\geq -1}$ and let $G_{\mathcal{K}}$ be the Galois group of an extension $\mathcal{K}$ of $\mathbf{Q}_p$ inside $\overline{\mathbf{Q}_p}$ such that $\mathcal{K}/\mathbf{Q}_p$ is finitely ramified. Let $I_{\mathcal{K}}$ be the inertia subgroup of $G_{\mathcal{K}}$. A Galois representation ρ of $G_{\mathcal{K}}$ is said to be *ramified of level i* if for all $u \in \mathbf{R}$ with $u > i$ the ramification group with upper numbering $I_{\mathcal{K}}^u$ is contained in $\ker(\rho)$.

We define the level of ramification for an abelian variety over K using the natural action of the absolute Galois group G_K on its ℓ -adic Tate module $T_{\ell}(A)$:

Definition 1.2. Let $i \in \mathbf{R}_{\geq -1}$. Let A be an abelian variety over $\mathcal{K}$ and let ℓ be a rational prime different than the residue field characteristic of $\mathcal{K}$. Then A is *ramified of level i* if $\rho_{A, \ell} : G_{\mathcal{K}} \rightarrow \text{Aut}(T_{\ell}(A))$ is ramified of level i .

Just as the conductor of an abelian variety A does not depend on the choice of ℓ in the definition of the representation $\rho_{A, \ell}$, so the level of ramification of A does not depend on the choice of ℓ . We will show this in Section 3.

In Section 4, we find all newforms f with level N a power of p , such that the level of ramification of A_f satisfies a certain bound. This bound is such that we are still able to use the tables of Odlyzko [Mar82]. For such newforms f , these tables then enable us to prove the modularity of an abelian variety over $\mathbf{Q}$ with good reduction outside p and with level of ramification at p at most the level of ramification of A_f at p . These newforms turn out to have level 27, 32 or 49. Using this information, we then prove the modularity of abelian varieties over $\mathbf{Q}$ with good reduction outside either 2, 3 or 7 that have the same level of ramification as the newforms of level resp. 32, 27 or 49:

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