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Asymptotic expansions for Barnes G-function

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ABSTRACT

We present two classes of asymptotic expansions for Barnes G-function, and provide the formulas for determining the coefficients of each class of the asymptotic expansions.
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The double gamma function Γ_2 and the multiple gamma functions Γ_n were introduced and investigated by Barnes in a series of papers [1–4]. Barnes applied these functions in the theory of elliptic functions and theta functions. Nonetheless, except possibly for the citations of Γ_2 in the exercises by Whittaker and Watson [27, p. 264] and also by Gradshteyn and Ryzhik [13, 13, p. 661, Entry 6.441 (4); p. 937, Entry 8.333], these functions were revived only in about the middle of the 1980s in the study of the determinants of the Laplacians on the n-dimensional unit sphere S^n (see, e.g., [11,16,18,21,25,26]). The

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theory of the double gamma function has indeed found interesting applications in many other recent investigations (see, for details, [23,24]).

We begin by recalling that Barnes G-function ($1/G = \Gamma_2$ being the so-called double gamma function) is defined as the integral function [1]:

$$\begin{aligned}
&\left[\Gamma_2(z+1)\right]^{-1} \\
&\quad = G(z+1) \\
&\quad = (2\pi)^{z/2}\exp\left(-\frac{1}{2}z-\frac{1}{2}(\gamma+1)z^2\right)\prod_{k=1}^{\infty}\left[\left(1+\frac{z}{k}\right)^k\cdot\exp\left(-z+\frac{z^2}{2k}\right)\right],
\end{aligned} \tag{1}$$

where $\gamma = 0.5772156649\ldots$ denotes the Euler–Mascheroni constant. Barnes G-function satisfies $G(1) = 1$ and $G(z+1) = \Gamma(z)G(z)$, where Γ denotes the gamma function.

The following integral representation for Barnes G-function was established by Ferreira and López [12, Theorem 1]: For $|\mathrm{Arg}(z)| < \pi$,

$$\begin{aligned}
\ln G(z+1) = {}& \frac{1}{4}z^2 + z\ln\Gamma(z+1) - \left(\frac{1}{2}z^2+\frac{1}{2}z+\frac{1}{12}\right)\ln z - \ln A \\
&+ \sum_{k=1}^{N-1}\frac{B_{2k+2}}{2k(2k+1)(2k+2)z^{2k}} + R_N(z) \quad (N = 1, 2, \ldots),
\end{aligned} \tag{2}$$

where B_{2k+2} are the Bernoulli numbers and A is the Glaisher–Kinkelin constant defined by

$$\ln A = \lim_{n\to\infty}\left\{\ln\left(\prod_{k=1}^{n}k^k\right) - \left(\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}\right)\ln n + \frac{n^2}{4}\right\}, \tag{3}$$

the numerical value of A being $1.282427\ldots$. The remainder $R_N(z)$ is for $\Re(z) > 0$ given by

$$R_N(z) = \int_0^\infty\left(\frac{t}{e^t-1} - \sum_{k=0}^{2N}\frac{B_k}{k!}t^k\right)\frac{e^{-zt}}{t^3}\,dt. \tag{4}$$

Estimates for $|R_N(z)|$ are also found by Ferreira and López [12], showing that the expansion is indeed an asymptotic expansion of $\ln G(z+1)$ in sectors of the complex plane cut along the negative axis. Pedersen [19, Theorem 1.1] proved that for any $N \geqslant 1$, the function $x \mapsto (-1)^N R_N(x)$ is completely monotonic on $(0, \infty)$. Other asymptotic relations (avoiding the $\ln\Gamma$ term) have been obtained by Ruijsenaars [22] and investigated by Pedersen [20], Koumandos [14] and Koumandos and Pedersen [15]. Recently, some upper and lower bounds for the double gamma function were derived in terms of the gamma, psi and polygamma functions, see [5–7,10]. Chen [8] and Mortici [17] established the inequalities and asymptotic expansions for $\ln A$ in (3). By using the Bell polynomials,

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