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Some divisibility properties of binomial and q-binomial coefficients

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ABSTRACT

We first prove that if a has a prime factor not dividing b then there are infinitely many positive integers n such that $\binom{an+bn}{an}$ is not divisible by $bn+1$. This confirms a recent conjecture of Z.-W. Sun. Moreover, we provide some new divisibility properties of binomial coefficients: for example, we prove that $\binom{12n}{3n}$ and $\binom{12n}{4n}$ are divisible by $6n-1$, and that $\binom{330n}{88n}$ is divisible by $66n-1$, for all positive integers n. As we show, the latter results are in fact consequences of divisibility and positivity results for quotients of q-binomial coefficients by q-integers, generalising the positivity of q-Catalan numbers. We also put forward several related conjectures.

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1. Introduction

The study of arithmetic properties of binomial coefficients has a long history. In 1819, Babbage [6] proved the congruence

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^2}$$

for primes $p \geqslant 3$. In 1862, Wolstenholme [28] showed that the above congruence holds modulo p^3 for any prime $p \geqslant 5$. See [20] for a historical survey on Wolstenholme's theorem. Another famous congruence is

$$\binom{2n}{n} \equiv 0 \pmod{n+1}.$$

The corresponding quotients, the numbers $C_n := \frac{1}{n+1}\binom{2n}{n}$, are called *Catalan numbers*, and they have many interesting combinatorial interpretations; see, for example, [12] and [24, pp. 219–229]. Recently, Ulas and Schinzel [27] studied divisibility problems of Erdős and Straus, and of Erdős and Graham. In [25,26], Sun gave some new divisibility properties of binomial coefficients and their products. For example, Sun proved the following result.

Theorem 1.1. *(See [26, Theorem 1.1].) Let a, b, and n be positive integers. Then*

$$\binom{an+bn}{an} \equiv 0 \mod \frac{bn+1}{\gcd(a, bn+1)}. \tag{1.1}$$

Sun also proposed the following conjecture.

Conjecture 1.2. *(See [26, Conjecture 1.1].) Let a and b be positive integers. If $(bn+1) \mid \binom{an+bn}{an}$ for all sufficiently large positive integers n, then each prime factor of a divides b. In other words, if a has a prime factor not dividing b, then there are infinitely many positive integers n such that $(bn+1) \nmid \binom{an+bn}{an}$.*

Inspired by Conjecture 1.2, Sun [26] introduced a new function $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{N}$. Namely, for positive integers a and b, if $\binom{an+bn}{an}$ is divisible by $bn+1$ for all $n \in \mathbb{Z}^+$, then he defined $f(a,b) = 0$; otherwise, he let $f(a,b)$ be the smallest positive integer n such that $\binom{an+bn}{an}$ is not divisible by $bn+1$. Using *Mathematica*, Sun [26] computed some values of the function f:

$$f(7,36) = 279, \qquad f(10,192) = 362, \qquad f(11,100) = 1187,$$
$$f(22,200) = 6462, \qquad \ldots.$$