Journal of Number Theory 135 (2014) 185–220

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Binary quadratic forms and the Fourier coefficients of certain weight 1 *eta*-quotients

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ARTICLE INFO

Article history:
Received 24 February 2013
Received in revised form 16 August 2013
Accepted 25 August 2013
Available online 23 October 2013
Communicated by David Goss

MSC:
11B65
11F03
11F11
11F20
11F27
11E16
11E20
11E25
14K25

Keywords:
Eta-quotients
Binary quadratic forms
Theta series
Multiplicative functions
Hecke operators
Dissections of q-series

ABSTRACT

We state and prove an identity which represents the most general *eta*-products of weight 1 by binary quadratic forms. We discuss the utility of binary quadratic forms in finding a multiplicative completion for certain *eta*-quotients. We then derive explicit formulas for the Fourier coefficients of certain *eta*-quotients of weight 1 and level 47, 71, 135, 648, 1024, and 1872.

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http://dx.doi.org/10.1016/j.jnt.2013.08.016

1. Introduction and notation

Throughout the paper we assume q is a complex number with $|q| < 1$. We use the standard notations

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n) \tag{1.1}$$

and

$$E(q) := (q;q)_\infty. \tag{1.2}$$

Next, we recall the Ramanujan theta function

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \quad |ab| < 1. \tag{1.3}$$

The function $f(a,b)$ satisfies the Jacobi triple product identity [3, Entry 19]

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \tag{1.4}$$

along with

$$f(a,b) = a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} f\big(a(ab)^n, b(ab)^{-n}\big), \tag{1.5}$$

where $n \in \mathbb{Z}$ [3, Entry 18]. One may use (1.4) to derive the following special cases:

$$E(q) = f\big(-q,-q^2\big) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}}, \tag{1.6}$$

$$\phi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{E^5(q^2)}{E^2(q^4)E^2(q)}, \tag{1.7}$$

$$\psi(q) := f\big(q,q^3\big) = \sum_{n=-\infty}^{\infty} q^{2n^2-n} = \frac{E^2(q^2)}{E(q)}, \tag{1.8}$$

$$f\big(q,q^2\big) = \frac{E^2(q^3)E(q^2)}{E(q^6)E(q)}, \tag{1.9}$$

$$f\big(q,q^5\big) = \frac{E(q^{12})E^2(q^2)E(q^3)}{E(q^6)E(q^4)E(q)}. \tag{1.10}$$

Note that (1.6) is the famous Euler pentagonal number theorem. Splitting (1.6) according to the parity of the index of summation, we find

$$E(q) = \sum_{n=-\infty}^{\infty} q^{6n^2-n} - q\sum_{n=-\infty}^{\infty} q^{6n^2+5n} = f\big(q^5,q^7\big) - qf\big(q,q^{11}\big). \tag{1.11}$$

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