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Differential modular forms on Shimura curves over totally real fields

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ABSTRACT

We study the theory of differential modular forms for compact Shimura curves over totally real fields and construct differential modular forms, which are generalizations of the fundamental differential modular forms. We also construct the Serre–Tate expansions of such differential modular forms as a possible alternative to the Fourier expansion maps and calculate the Serre–Tate expansions of some of these differential modular forms.

1. Introduction

The differential modular forms are invented by Buium and his collaborators in a beautiful series of papers [8–10,12,11]. These are the modular forms obtained by applying the arithmetic p-jet space functor (adjoint to the p-typical Witt vector functor) to the ring of modular forms. We wish to understand the differential modular forms obtained by applying arithmetic jet space functor to the ring of modular forms on Shimura curves over totally real fields [19]. This paper is a modest attempt to study the differential modular forms for Shimura curves over totally real fields extending the results for the Shimura curves over $\mathbb{Q}$ [9,10]. We note that the present paper is the first initiative to investigate the differential modular forms for fields different from $\mathbb{Q}$. It is expected that the study of the differential modular forms on the Shimura curves over totally real fields should be

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useful in an "effective" proof of the André–Oort conjecture for the Shimura curves over totally real fields. The aim of the present paper is to study the following questions.

Question 1. Describe the Shimura curves over totally real fields modulo Hecke correspondences.

Question 2. Describe the quotient of Shimura curves over totally real fields modulo isogeny.

Question 3. Describe the "test objects" of the Shimura curves over totally real fields which have lifts of Frobenius.

Question 4. Describe explicit lifts of Hasse invariants for the Shimura curves over totally real fields.

We observe that isogeny covariance is stronger condition than being Hecke equivariance. Inspired by Kolchin's theory for differential algebras, Buium introduced δ-geometry and the theory of differential modular forms to answer these questions. δ is an analogue of the differentials for number fields. We enlarge the algebraic geometry to δ-geometry of Buium to study the questions. As in [11], we replace polynomials with arithmetic differential equations to describe the *categorical* quotients of Questions 1, 2 and the geometrically significant class of abelian schemes inside the unitary PEL Shimura curves as in Questions 3 and 4.

Some of the differential modular forms over totally real fields also have certain symmetries, namely isogeny covariance. This is one of the motivation of the study/construction of the differential modular forms. We closely follow the constructions of basic differential modular forms of Buium to show that there exist differential modular forms over totally real fields, whose zero sets are the solutions to the questions.

Let $f = \sum_n a_n q^n$ be a classical elliptic newform of weight 2 with respect to the congruence subgroup $\Gamma_1(N)$. Let $K_f = \mathbb{Q}(\{a_n\}_n)$ be the coefficient field of this newform with $g_f = [K_f : \mathbb{Q}]$. Buium attached g_f differential eigenforms of order 2, weight 0 to such a classical newform in [12]. We fix a totally real number field F with ring of integers O_F and let N be an ideal of O_F. We assume that the field F and the ideal N satisfy the following *Jacquet–Langlands* condition: either $[F : \mathbb{Q}]$ is odd or $\mathrm{ord}_v(N) = 1$ for all $v \mid N$. Let $\overline{T}$ denote $T \otimes \prod_p \mathbb{Z}_p$ for any abelian group T and let $K_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\overline{O_F}) | \overline{N} \mid c\}$ be a subgroup of $\mathrm{GL}_2(\overline{F})$.

Let p be an odd prime not dividing the discriminant of the totally real field F and let $\mathfrak{P}$ be a prime ideal of O_F over p. We also assume that a fixed quaternion algebra over F is split at $\mathfrak{P}$ (cf. Section 2). In this paper, we will study differential modular forms w.r.t. this fixed prime ideal.

Let $\mathbf{f}$ be a Hilbert modular newform over F of parallel weight 2, level $K_0(N)$ and trivial central character with the coefficient field $K_{\mathbf{f}}$, a number field of degree h. Then there is an abelian variety of dimension h with a motivic L-function. This abelian variety