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Explicit upper bounds for the Stieltjes constants

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ABSTRACT

Text. Let χ be a primitive Dirichlet character modulo q and let $(-1)^n\gamma_n(\chi)/n!$ (for n larger than 0) be the n-th Laurent coefficient around $z=1$ of the associated Dirichlet L-series. When χ is non-principal, $(-1)^n\gamma_n(\chi)$ is simply the value of the n-th derivative of $L(z,\chi)$ at $z=1$. In this paper we give an explicit upper bounds for $|\gamma_n(\chi)|$ for $q \leqslant \frac{\pi}{2}\frac{e^{(n+1)/2}}{n+1}$. In particular, when $q=1$ the explicit upper bound we get improves on earlier work. We conclude this paper by showing that we can altogether dispense in these proofs with the functional equation of $L(z,\chi)$.

Video. For a video summary of this paper, please click here or visit http://www.youtube.com/watch?v=q340UciEvAA.

1. Introduction and results

Let χ be a Dirichlet character modulo q and let us denote by $\gamma_n(\chi)$ (for n larger than 0) the n-th Laurent coefficient around $z=1$ of the Dirichlet L-function $L(z,\chi)$. We quote the following relation from [9],

$$\gamma_n(\chi) = \sum_{a=1}^{q} \chi(a)\gamma_n(a,q), \tag{1}$$

with

$$\gamma_n(a,q) = \lim_{T\to\infty}\left\{ \sum_{1\leqslant m\equiv a \bmod q}^{T} \frac{(\log m)^n}{m} - \frac{(\log T)^{n+1}}{q(n+1)} \right\}. \tag{2}$$

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In particular, $\gamma_0(1,1)$ is the well-known Euler constant. The constants $\gamma_n(a,q)$ are known as the Stieltjes or the generalized Euler constants. We use the shorter form $\gamma_n(1,1)=\gamma_n$ (this is also $\gamma_n(\chi_0)$ where χ_0 is the only character modulo 1), so that we have

$$\zeta(z)=\frac{1}{z-1}+\sum_{n=0}^{\infty}(-1)^n\frac{\gamma_n}{n!}(z-1)^n.$$

The problem of finding an explicit upper bound for $|\gamma_n|$ has been addressed by a number of authors. Briggs [2] started this line of investigation by proving that

$$|\gamma_n|<\left(\frac{n}{2e}\right)^n.$$

In 1985, Matsuoka [12] produced an asymptotic expression for γ_n, and he was also able to simplify his method to derive the explicit form:

$$\forall n\geqslant 10,\quad |\gamma_n|\leqslant 10^{-4}e^{n\log\log n}. \tag{3}$$

In the paper [10], Kreminski conjectured on numerical evidence (see Table 2) that the above inequality may be considerably strengthened. Recently, Adell [1] proved that, for any $n\geqslant 4$, we have

$$|\gamma_n|\leqslant\left(\frac{n!e^m}{m^{n+1}}\left(\frac{n+1}{m}+1\right)+\frac{1}{n+1}\right)\log^{n+1}(m+1),$$

where $m=\lfloor n(1-1/\log n)\rfloor$ and $\lfloor x\rfloor$ denotes the integer part of x.

Let us turn to the corresponding problem with a character. In 1994, Toyoizumi [13] studied the problem of bounding $|\gamma_n(\chi)|$ when n is fixed and q goes to infinity. On using Burgess inequality, he showed that for real non-principal χ, when q is cube-free, for any $\epsilon>0$, we have

$$\left|L^{(n)}(1,\chi)\right|\leqslant\left(\frac{1}{(n+1)4^{n+1}}\cdot\frac{L(1+\epsilon,\chi)}{\zeta(1+\epsilon)}+\epsilon\right)\log^{n+1}q, \tag{4}$$

when $q>q_0(\epsilon)$, where $q_0(\epsilon)$ is a constant depending only on ϵ.

In another direction and pursuing the groundbreaking result of Matsuoka [12], Ishikawa [6] studied the asymptotic behavior of $L^{(n)}(1,\chi)$ as $n\to\infty$. He showed that there exists an n_0 such that for all $n\geqslant n_0$,

$$\left|L^{(n)}(1,\chi)\right|\leqslant q^{n/\log n-1/2}\exp\left\{n\log\log n-\frac{n\log\log n}{\log n}\right\}. \tag{5}$$

We note here that the latter result is better than Eq. (4) when q is small with respect to n.

In this paper, we produce an explicit upper bound of $|\gamma_n(\chi)|$ useful for large values of n.

Theorem 1. *Let χ be a primitive Dirichlet character to modulus q. Then, for every $1\leqslant q\leqslant\frac{\pi}{2}\frac{e^{(n+1)/2}}{n+1}$, we have*

$$\frac{|\gamma_n(\chi)|}{n!}\leqslant q^{-1/2}C(n,q)\big(1+D(n,q)\big)$$

with