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On the asymptotic density of the support of a Dirichlet convolution

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ABSTRACT

Let ν be a multiplicative arithmetic function with support of positive asymptotic density. We prove that for any not identically zero arithmetic function f such that $\sum_{f(n)\neq 0} 1/n < \infty$, the support of the Dirichlet convolution $f * \nu$ possesses a positive asymptotic density. When f is a multiplicative function, we give also a quantitative version of this claim. This generalizes a previous result of P. Pollack and the author, concerning the support of Möbius and Dirichlet transforms of arithmetic functions.

1. Introduction

Let f and g be two arithmetic functions, i.e., functions from the set of positive integers to the set of complex numbers. The Dirichlet convolution of f and g is the arithmetic function denoted by $f * g$ and defined as

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$$(f * g)(n) := \sum_{d \mid n} f(d) g(n/d),$$

for all positive integers n. The set of arithmetic functions together with Dirichlet convolution forms a commutative monoid with identity element ϵ, the arithmetic function that satisfies $\epsilon(1) = 1$ and $\epsilon(n) = 0$ for all integers $n \geqslant 2$. Furthermore, any arithmetic function f has an inverse f^{-1} with respect to the Dirichlet convolution if and only if $f(1) \neq 0$, in which case f^{-1} can be computed recursively by the identities $f^{-1}(1) = 1/f(1)$ and

$$f^{-1}(n) = -\frac{1}{f(1)} \sum_{\substack{d \mid n \\ d < n}} f^{-1}(d) f(n/d), \quad n \geqslant 2.$$

The Dirichlet transform $\hat{f}$ and the Möbius transform $\check{f}$ of the arithmetic function f are defined by $\hat{f} := f * 1$ and $\check{f} := f * \mu$, where μ is the Möbius function. Notably, the Dirichlet inverse of μ is the identically equal to 1 arithmetic function. Actually, this is the content of the well-know Möbius inversion formula, that is $\check{\hat{f}} = \hat{\check{f}} = f$ (see [Hua09, Ch. 2] for details).

We call (f, g) a Möbius pair if f and g are arithmetic function with $f = \hat{g}$, or equivalently $g = \check{f}$. In a previous paper, P. Pollack and the author studied the asymptotic density of the support of functions f and g in a Möbius pair (f, g); by the support of an arithmetic function h we mean the set of all positive integers n such that $h(n) \neq 0$, we denote it with $\operatorname{supp}(h)$. They give the following result [PS13, Theorem 1.1].

Theorem 1.1. *Suppose that (f, g) is a nonzero Möbius pair. If $\operatorname{supp}(f)$ is thin then $\operatorname{supp}(g)$ possesses a positive asymptotic density. The same result holds with the roles of f and g reversed.*

We recall that a set of positive integers $\mathscr{A}$ is said to be *thin* if $\sum_{a \in \mathscr{A}} 1/a < \infty$. Given a multiplicative arithmetic function ν, we extend the notion of Möbius pair by saying that (f, g) is a ν-pair if f and g are arithmetic function with $g = f * \nu$, or equivalently $f = g * \nu^{-1}$. (Note that ν is Dirichlet invertible since as a multiplicative function it satisfies $\nu(1) = 1$.) Here, we prove the following generalization of Theorem 1.1.

Theorem 1.2. *Let ν be a multiplicative arithmetic function with support of positive asymptotic density. Suppose that (f, g) is a nonzero ν-pair. If $\operatorname{supp}(f)$ is thin then $\operatorname{supp}(g)$ possesses a positive asymptotic density.*

This is a true generalization of Theorem 1.1, since 1 and μ are multiplicative arithmetic functions with support of positive asymptotic density. Precisely, $\operatorname{supp}(\mu)$ is the set of squarefree numbers and it has density $6/\pi^2$, as is well known.

The discovery of Theorem 1.1 was initially motivated by the desire to prove a kind of *uncertainty principle for the Möbius transform*, in the sense that f and g cannot both