Journal of Number Theory 134 (2014) 13–28

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Integral cohomology of certain Picard modular surfaces

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ARTICLE INFO

Article history:
Received 7 September 2007
Received in revised form 8 July 2013
Accepted 8 July 2013
Available online 12 September 2013
Communicated by David Goss

MSC:
primary 11F75

Keywords:
Picard modular group
Locally symmetric space
Cohomology of arithmetic subgroups
Spine

ABSTRACT

Let $\bar{\Gamma}$ be the Picard modular group of an imaginary quadratic number field k and let $\mathcal{D}$ be the associated symmetric space. Let $\Gamma \subset \bar{\Gamma}$ be a congruence subgroup. We describe a method to compute the integral cohomology of the locally symmetric space $\Gamma\backslash\mathcal{D}$. The method is implemented for the cases $k = \mathbb{Q}(i)$ and $k = \mathbb{Q}(\sqrt{-3})$, and the cohomology is computed for various Γ.

1. Introduction

Let $\bar{\Gamma} = \mathrm{SU}(2,1;\mathcal{O}_k)$ be the Picard modular group of an imaginary quadratic number field k and let $\mathcal{D}$ be the associated symmetric space. Let $\Gamma \subset \bar{\Gamma}$ be a congruence subgroup. Although $\mathcal{D}$ is 4-dimensional, the virtual cohomological dimension of $\bar{\Gamma}$ is 3. Hence the cohomology $H^i(\Gamma\backslash\mathcal{D})$ vanishes for $i > 3$.

If Γ is torsion-free, then $\Gamma\backslash\mathcal{D}$ is an Eilenberg–MacLane space for Γ. It follows that the group cohomology of Γ with trivial complex coefficients is isomorphic to the complex cohomology of the locally symmetric space,

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0022-314X/$ – see front matter
http://dx.doi.org/10.1016/j.jnt.2013.07.015

$$H^*(\Gamma;\mathbb{C}) \simeq H^*(\Gamma\backslash\mathcal{D};\mathbb{C}). \tag{1}$$

In fact, (1) remains true while using complex coefficients when Γ has torsion, but is not true in general for integral coefficients.

A deep result of Franke [7] describes a relationship between cohomology and automorphic forms. The cohomology groups $H^*(\Gamma;\mathbb{C})$, or more generally $H^*(\Gamma;M)$ for any complex finite dimensional rational representation of $\mathrm{SU}(2,1)$, decompose into cuspidal cohomology and Eisenstein cohomology. The cuspidal cohomology of Γ can be represented by cuspidal automorphic forms. It is possible to compute the space of cuspidal Picard modular forms with the Hecke action using the Jacquet–Langlands correspondence [5]. However, these methods will not compute the torsion classes in the integral cohomology of the group or the locally symmetric space. In this paper, we use topological methods to compute the torsion classes in the integral cohomology of the locally symmetric space $\Gamma\backslash\mathcal{D}$.

There is a 3-dimensional cell complex W, known as a spine, that can be used to compute the integral cohomology of the locally symmetric space $\Gamma\backslash\mathcal{D}$. The existence of such a W for general $\mathbb{Q}$-rank 1 groups is known [9], but there are few explicit examples for non-linear symmetric spaces. We outline a method of computing W for $\mathrm{SU}(2,1;\mathcal{O}_k)$, where k is an imaginary quadratic number field with class number 1. The structure of the spine is computed in [10] for the Gaussian case $k=\mathbb{Q}(i)$ and is computed here for the Eisenstein case $k=\mathbb{Q}(\sqrt{-3})$. Falbel and Parker [6] do similar computations for the Eisenstein–Picard modular group, but with a different purpose. They exhibit fundamental domain for the action of $\bar{\Gamma}$ on $\mathcal{D}$. Using the structure and combinatorics of the fundamental domain, they deduce a presentation for $\bar{\Gamma}$.

The outline of the paper is as follows. We first recall the Picard modular group and associated symmetric space in Section 2. A method of computing the spine W is given in Section 3. Section 4 outlines the method of [2] and its implications in the context of our cell complex. The cell complex is computed in [10] for the Picard modular group over the Gaussian integers. We compute the cell complex and give the stabilizers for Picard modular group over the Eisenstein integers in Section 5. Finally, Section 6 gives the cohomology computation results.

2. Preliminaries

Let k be an imaginary quadratic field with discriminant D and ring of integers $\mathcal{O}$. Thus $k=\mathbb{Q}(\sqrt{D})$ and D is either square-free and $D \equiv 1 \bmod 4$ or $D=4D'$, where D' is square-free and $D' \equiv 2 \bmod 4$ or $D' \equiv 3 \bmod 4$. Then $\mathcal{O}$ is generated by 1 and $\omega=(D+\sqrt{D})/2$. For $d=1,2,3,7,11,19,43,67,163$, $\mathbb{Q}(\sqrt{-d})$ has class number $h(k)=1$ and $\mathcal{O}$ is a principal ideal domain. Fix an imaginary quadratic field k with class number 1.