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On bases of Washington's group of circular units of some real cyclic number fields

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ABSTRACT

Let K be a cyclic field whose genus field in the narrow sense is real and which is totally ramified at each ramifying prime. We shall find a basis for the group $C_W(K)$ of circular units of K defined by Washington. This enables us to compute the index of $C_W(K)$ in the group of all units of K.

1. Introduction

For any positive integer r let ζ_r be a primitive rth root of unity, e.g. $\zeta_r = e^{2\pi i/r}$. By an abelian field we have in mind a finite Galois extension of $\mathbb{Q}$ with an abelian Galois group. According to the Kronecker–Weber theorem we know that any abelian field is a subfield of a suitable cyclotomic field $\mathbb{Q}(\zeta_n)$. For an abelian field F of conductor n (i.e. n is the smallest positive integer such that F is a subfield of $\mathbb{Q}(\zeta_n)$) let $E(F)$ be the

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group of all units of the ring of integers of F. In [13] Sinnott introduced the group $\mathcal{C}(F)$ of circular numbers of F which can be defined by

$$\mathcal{C}(F) = \langle -1, \mathrm{N}_{\mathbb{Q}(\zeta_r)/\mathbb{Q}(\zeta_r)\cap F}\bigl(1-\zeta_r^a\bigr);\ r, a \in \mathbb{Z},\ 1 < r \bigm| n, \gcd(a,r) = 1\rangle,$$

where $\langle \ldots \rangle$ means "generated as a multiplicative subgroup of $F^\times$", and also the group $C_S(F)$ of circular units of F by

$$C_S(F) = E(F) \cap \mathcal{C}(F).$$

In fact, instead of $\mathcal{C}(F)$ Sinnott used a larger group considering all integers $r > 1$, but this group has the same intersection with $E(F)$ (see [12]). He also found an index formula which relates the index $[E(F) : C_S(F)]$ and the class number h_F^+ of the maximal real subfield $F \cap \mathbb{R}$ of F but this formula unfortunately contains a non-explicit factor which has been computed only in some special cases; we shall deal with two of them in Remark 4 and in Proposition 7.

Another definition of the group of the circular units of F was suggested by Washington in [14, p. 143]:

$$C_W(F) = F \cap C_S\bigl(\mathbb{Q}(\zeta_n)\bigr).$$

Trivially $C_S(F) \subseteq C_W(F) \subseteq E(F)$, therefore, $C_W(F)$ is a better approximation of $E(F)$ than $C_S(F)$. Sinnott's group $C_S(F)$ is the one that is used most often now. Not much seems to be known about Washington's group $C_W(F)$ apart from [4] computing the index $[E(F) : C_W(F)]$ for some composita of quadratic fields. There are papers [10,1,11,3,2] studying some properties of $C_W(F_n)$ in the cyclotomic $\mathbb{Z}_p$-extension $\bigcup_{n=0}^{\infty} F_n$ of an abelian field F_0; for example, it is known that $\varprojlim C_W(F)/C_S(F)$ is the maximal finite Λ-submodule of $\varprojlim E(F)/C_S(F)$. But $C_W(F)$ deserves more attention, for example the knowledge of an explicit unit in $C_W(F) - C_S(F)$ can be used in Rubin's machinery to produce annihilators of the class group of F (see [7] and [8] for the notion of "semispecial units").

This paper is another attempt to say more about $C_W(F)$. We shall generalize the result of [4] to any real genus field in the narrow sense of an abelian field (see Remark 4 below). But the main aim of this paper is to study $C_W(K)$ for a cyclic field K (i.e. an absolutely abelian field with a cyclic Galois group) which is totally ramified at each ramifying prime and whose genus field in the narrow sense is real. Namely, in Proposition 10 we shall describe a $\mathbb{Z}$-basis of $C_W(K)$, then we shall compute the index $[C_W(K) : C_S(K)]$ and from this and from Sinnott's index formula we obtain an explicit formula for the index $[E(K) : C_W(K)]$ containing the class number h_K of the field K (see Theorem 11). As a consequence we get a non-trivial divisibility relation for h_K.

2. Preliminaries and notations

Let $s \geqslant 2$, $I = \{1, \ldots, s\}$, $p_1, \ldots, p_s$ be distinct primes, q_i a power of p_i for $i \in I$, ζ_i be a fixed q_ith primitive root of unity for each $i \in I$ (from now on we shall use this new