Journal of Number Theory 134 (2014) 142–180

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Congruences to Ikeda–Miyawaki lifts and triple L-values of elliptic modular forms ☆

Tomoyoshi Ibukiyama [a], Hidenori Katsurada [b,*], Cris Poor [c], David S. Yuen [d]

[a] *Department of Mathematics, Graduate School of Science, Osaka University, Machikaneyama 1-1, Toyonaka, Osaka 560-0043, Japan*
[b] *Muroran Institute of Technology, Mizumoto 27-1, Muroran 050-8585, Japan*
[c] *Department of Mathematics, Fordham University, Bronx, NY 10458, USA*
[d] *Lake Forest College, 555 N. Sheridan Rd., Lake Forest, IL 60045, USA*

ARTICLE INFO

Article history:
Received 14 May 2013
Accepted 5 July 2013
Available online 19 September 2013
Communicated by D. Zagier

MSC:
11F46
11F67

Keywords:
Ikeda–Miyawaki lift
Congruences
Triple product L-function

ABSTRACT

In this paper, we consider congruences between the Ikeda–Miyawaki lift and other Siegel modular forms, relating these congruences to critical values of L-functions by using Ikeda's conjecture on periods. We also give general formulas for critical values of triple L-functions and prove results, both in theory and in examples, on the relation between such congruences and critical values.

1. Introduction

Congruences between modular forms are important in the arithmetic theory of modular forms. In particular, congruences between lifts and non-lifts sometimes produce

☆ The first two authors are partially supported by Grant-in-Aid for Scientific Research (No. 21244001), Japan Society for the Promotion of Science.

* Corresponding author.
E-mail addresses: ibukiyam@math.sci.osaka-u.ac.jp (T. Ibukiyama), hidenori@mmm.muroran-it.ac.jp (H. Katsurada), poor@fordham.edu (C. Poor), yuen@lakeforest.edu (D.S. Yuen).

0022-314X/$ – see front matter
http://dx.doi.org/10.1016/j.jnt.2013.07.013

nontrivial elements of the Bloch–Kato Selmer group (cf. [Br,BDS,DIK]). In [Kat5], the second-named author considered the congruence between the Duke–Imamoḡlu–Ikeda lift $I_{2n}(h)$ of a Hecke eigenform h of half-integral weight and non-Duke–Imamoḡlu–Ikeda lifts, and proved that a prime ideal dividing a certain L-value of f gives such a congruence, where f is the primitive form of integral weight corresponding to h under the Shimura correspondence. This result is based on the relation between the periods of $I_{2n}(h)$ and h proved by the second-named author and Kawamura [KK], which forms a part of the relations conjectured by Ikeda [Ik2]. A similar result concerning congruences between Yoshida lifts and non-Yoshida lifts was proved by Böcherer, Dummigan, and Schulze-Pillot [BDS]; this proof is also based on period relations, this time for the Yoshida lift. In general, the algebraic part of critical values of the standard L-function sometimes gives congruence primes between Siegel modular forms, see [Kat3]. In view of the above results, we can expect that if there is a formula to describe the period of a lift F from some form G by that of G, then the critical values of some L-function of G are related to congruences between the lift F and non-lifts.

In this article, we consider congruences between Ikeda–Miyawaki lifts and other Siegel modular Hecke eigenforms. Let k and n be positive integers such that $k+n+1$ is even. For a Hecke eigenform h of weight $k+1/2$ for $\Gamma_0(4)$ and a primitive form g of weight $k+n+1$ for $\mathrm{SL}_2(\mathbf{Z})$, let $\mathcal{F}_{h,g}$ be the cusp form of weight $k+n+1$ for $\mathrm{Sp}_{2n+1}(\mathbf{Z})$ constructed by Ikeda [Ik2]. For the precise definition of $\mathcal{F}_{h,g}$ see Section 3. This type of lift was conjectured by Miyawaki [Miy] in the case $n=1$, therefore we call $\mathcal{F}_{h,g}$ the Ikeda–Miyawaki lift of h and g. We also denote by f the primitive form of weight $2k$ for $\mathrm{SL}_2(\mathbf{Z})$ corresponding to h under the Shimura correspondence. Then, roughly speaking, our conjecture can be stated as follows (more precisely, see Conjecture B and Problem B′):

Let $\mathfrak{P}$ be a "big prime ideal" in the composite $\mathbf{Q}(f)\mathbf{Q}(g)$ of the Hecke fields of f and g. Then $\mathfrak{P}$ divides the algebraic part $L_{alg}(2k+2n, f\otimes g\otimes g)$ of the triple product L-function at $2k+2n$ if and only if there exists a congruence modulo $\mathfrak{P}$ between $\mathcal{F}_{h,g}$ and a cuspidal Hecke eigenform G of the same weight, where G is a non-Ikeda–Miyawaki lift.

This type of conjecture has already been proposed in the case of the Saito–Kurokawa lift and the Duke–Imamoḡlu–Ikeda lift (cf. [Kat2]). Our conjecture is based on the conjecture concerning the period of the Ikeda–Miyawaki lift proposed by Ikeda [Ik2] (cf. Conjecture A, Theorem 3.1 and its corollary). We note that Bergström, Faber, and van der Geer [BFG] have proposed a conjecture on the congruence of (not necessarily scalar valued) modular forms of degree three from a different point of view. We discuss the relation between their conjecture and ours in Section 3. Since certain types of triple L-functions appear in the description of the congruence primes, we give a concrete general formula for the special values of any triple L-function in the *balanced* case, and execute this calculation to give values in several cases, including the cases which appear as examples in the conjecture of [BFG]. Finally we construct examples of non-Ikeda–