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An improved upper bound for the argument of the Riemann zeta-function on the critical line II

Timothy S. Trudgian [1]

Mathematical Sciences Institute, The Australian National University, Australia

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ABSTRACT

Text. This paper concerns the function $S(T)$, where $\pi S(T)$ is the argument of the Riemann zeta-function along the critical line. The main result is that

$$\left|S(T)\right| \leqslant 0.112 \log T + 0.278 \log\log T + 2.510,$$

which holds for all $T \geqslant e$.

Video. For a video summary of this paper, please click here or visit http://youtu.be/FldP0idE0aI.

1. Summary of results

This paper is the sequel to [15] and is related to [16]; reference will be made frequently to these papers. Write

$$\left|S(T)\right| \leqslant a \log T + b \log\log T + c, \quad \text{for } T \geqslant T_0, \tag{1.1}$$

whence Table 1 provides a brief historical summary.

E-mail address: timothy.trudgian@anu.edu.au.
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Table 1
Bounds on $S(T)$ in (1.1).

	a	b	c	T_0
Von Mangoldt [20] 1905	0.432	1.917	12.204	28.588
Grossmann [6] 1913	0.291	1.787	6.137	50
Backlund [1] 1914	0.275	0.979	7.446	200
Backlund [2] 1918	0.137	0.443	4.35	200
Rosser [11] 1939	1.12	0	9.5	1450
Rosser [12] 1941	0.137	0.443	1.588	1467
Trudgian [15] 2012	0.17	0	1.998	e
Trudgian (Theorem 1) 2012	0.112	0.278	2.510	e

Note that the result in [15] improves on that in [12] when $25 \leqslant T \leqslant 10^{15}$. The purpose of this article is to improve on the result in [12] for all T. This is achieved with the following theorem.

Theorem 1. *If $T \geqslant e$, then*

$$\left|S(T)\right| \leqslant 0.112 \log T + 0.278 \log\log T + 2.510.$$

This implies the following result concerning $N(T)$, the number of complex zeroes of $\zeta(s)$ with imaginary parts in $(0, T)$.

Corollary 1. *If $T \geqslant e$, then*

$$\left|N(T) - \frac{T}{2\pi}\log\frac{T}{2\pi e} - \frac{7}{8}\right| \leqslant 0.112 \log T + 0.278 \log\log T + 2.510 + \frac{0.2}{T}.$$

It is known (see, e.g., [3,13,4,18,19]) that[2]

$$\begin{aligned} \left|S(T)\right| &\leqslant 1, \quad \text{for } 0 \leqslant T \leqslant 280, \\ \left|S(T)\right| &\leqslant 2, \quad \text{for } 0 \leqslant T \leqslant 6.8 \times 10^6, \\ \left|S(T)\right| &\leqslant 2.31366, \quad \text{for } 0 \leqslant T \leqslant 5.45 \times 10^8. \end{aligned} \tag{1.2}$$

The approach taken in this paper is to prove results initially for $T > T_0 > 5.45 \times 10^8$, and then for all T using (1.2). Theorem 1 is sharper than Rosser's bound in [12] whenever $T \geqslant 25$; for smaller values of T one is better placed using (1.2), which is superior to both Theorem 1 and the bound in [12].

Explicit bounds on $S(T)$ are used in calculations concerning the zeros of the zeta-function — see, e.g., [11,12]. Hence there is some interest in obtaining, not necessary the smallest coefficient of $\log T$ in Theorem 1, but good bounds of the form $|S(T)| \leqslant \alpha \log T$ for all $T \geqslant T_0$. This is accomplished by choosing appropriate values of parameters η and r that appear in the bound for $S(T)$ in Section 6. The results are summarised in Table 2.

[2] Indeed, the first two lines in (1.2) are equivalent to Gram's Law holding for all $0 \leqslant T \leqslant 280$ and Rosser's Rule holding for all $0 \leqslant T \leqslant 6.8 \times 10^6$.