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Change of root numbers of elliptic curves under extension of scalars ☆

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ABSTRACT

In this paper we study how the root number attached to an elliptic curve E over a finite field extension K of $\mathbb{Q}_3$ changes when E is considered as an elliptic curve over a finite Galois extension F of K via extension of scalars. The main result is a formula relating the root number $W(E/F)$ attached to $E \times_K F$ to the root number $W(E/K)$ attached to E.

0. Introduction

Let K be a finite field extension of $\mathbb{Q}_p$ with a fixed algebraic closure $\overline{K}$ and let $F \subset \overline{K}$ be a finite field extension of K. The main goal of the paper is to relate the root number $W(E/K)$ attached to an elliptic curve E over K to the root number $W(E/F)$ attached to elliptic curve $E \times_K F$ over F obtained from E via extension of scalars.

Explicit formulas for $W(E/K)$ in terms of the coefficients of an arbitrary generalized Weierstrass equation of E have been obtained by D. Rohrlich [6] in the case when

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E has potential multiplicative reduction over K and under the additional assumption $p \geqslant 5$ in the case when E has potential good reduction over K. Thus Rohrlich's formulas can be used to calculate $W(E/F)$ using an arbitrary Weierstrass equation of E over K. In the case $p = 3$ formulas for $W(E/K)$ were obtained by S. Kobayashi [4] in terms of the coefficients of a minimal Weierstrass equation of E over K, so in order to apply Kobayashi's formulas to calculate $W(E/F)$ one needs to find a minimal Weierstrass equation of E over F. Our motivation is to calculate $W(E/F)$ using a Weierstrass equation of E over K. The cases $p = 2$ or 3, E has potential good reduction over K, and F is an arbitrary finite field extension of K still remain untreated in full generality. We answer the question when $p = 3$ under an additional assumption that F is Galois over K.

Assume E has potential good reduction over K and $F \subset \overline{K}$ is a finite field extension of K. By definition, the root number $W(E/K)$ is the root number of representation σ_E of the Weil group $\mathcal{W}(\overline{K}/K)$ of K attached to E. It is known that σ_E is a two-dimensional semisimple representation of $\mathcal{W}(\overline{K}/K)$. If σ_E is not irreducible, then one can easily deduce from well-known formulas that

$$W(E/F) = W(E/K)^{[F:K]}$$

(see e.g. [6, p. 128]).

If σ_E is irreducible and p is odd (i.e., $p \neq 2$), then σ_E is induced by a multiplicative character of a quadratic extension $H \subset \overline{K}$ of K. Moreover, E has the Kodaira–Néron type III, III^*, II, IV, IV^*, or II^* (see Proposition 1.6 below). Furthermore,

- $H = K(\sqrt{-1}\,)$ if E is of type III or III^*,
- $H = K(\Delta^{1/2})$ if E is of type II, IV, IV^*, or II^*, where Δ is a discriminant of E.

The main results of the paper together with easy cases, which we include for the sake of completeness, can be summarized in the following

Theorem. *Let $F \subset \overline{K}$ be a finite field extension of K with ramification index $e(F/K)$ over K. Suppose p is odd, E has potential good reduction over K, and σ_E is irreducible.*

- *If $H \subseteq F$, then*

$$W(E/F) = \left(\frac{-1}{\hat{K}}\right)^{\delta}, \quad \delta = \begin{cases} \frac{[F:K]}{2}, & \text{if } H/K \text{ ramified}, \\ 0, & \text{if } H/K \text{ unramified}, \end{cases}$$

where $\hat{K}$ denotes the residue field of K and $(\frac{x}{\hat{K}})$ is the quadratic residue symbol of $x \in \hat{K}$ (Lemma 2.1 below).

- *If $H \not\subseteq F$, $p \geqslant 5$, then*

$$W(E/F) = (-1)^{\alpha + [F:K]} W(E/K)^{[F:K]},$$

where