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A general estimation for partitions with large difference: For even case

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ABSTRACT

Let $x_{N,i}(n)$ denote the number of partitions of n with difference at least N and minimal component at least i, and $y_{M,j}(n)$ the number of partitions of n into parts which are $\pm j \pmod M$. If N is even and i is co-prime with $N+2i+1$, we prove that

$$x_{N,i}(n) \geqslant y_{N+2i+1,i}(n)$$

for any positive integer n. This result partially generalizes the Alder–Andrews conjecture. Moreover, we also prove that

$$y_{\nu,1}(n) \geqslant y_{\nu,i}(n)$$

for any n if $i < \nu/2$ is co-prime with ν.

1. Introduction

The identities involving the infinity-sum and the infinity-product are always interesting and important. More than one hundred years ago, L.J. Rogers [12] first proved the following identities for $a = 0, 1$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+a)}}{(q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1-a})(1-q^{5n-4+a})} \tag{1}$$

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with

$$(q)_n = \prod_{j=1}^{n} \left(1 - q^j\right), \quad (q)_0 = 1. \tag{2}$$

Then these identities were proved by Rogers and Ramanujan [13] and by Schur [14] independently.

For any non-negative integer N and positive integer i, let $x_{N,i}(n)$ denote the number of partitions of n, whose parts have difference at least N and minimal component at least i. For positive integers $M \geqslant 3$ and j $(1 \leqslant j < M/2)$, let $y_{M,j}(n)$ denote the number of partitions of n into parts which are $\pm j \pmod M$.

As a generalization of (1) for $a = 0$, in 1956, H.L. Alder gave the following conjecture:

If $\Delta_d(n) = x_{d,1}(n) - y_{d+3,1}(n)$, then, for any $d, n \geqslant 1$, we have that $\Delta_d(n) \geqslant 0$.

In 1971, Andrews [2] proved Alder's conjecture for $d = 2^s - 1$ and $s \geqslant 4$, then he refined it as follows:

For $4 \leqslant d \leqslant 7$ and $n \geqslant 2d + 9$, or $d \geqslant 8$ and $n \geqslant d + 6$, $\Delta_d(n) \geqslant 0$.

In 2008, Yee [15] proved Alder's conjecture for $d = 7$ and $d \geqslant 32$.

In [1], the Andrews's refinement is completed.

Some generalizations in different forms are proved in [3,5] and others. These identities are very important for the vertex representations of affine Lie algebras, some linear basis of vacuum spaces or highest weight spaces can be given by them (Refs. [7,9–11]). Some other results also were proved in [4,6,8].

In this paper, we consider a partial generalization of the first Rogers–Ramanujan identity (also a partial generalization of Alder's conjecture). The main result is stated by the following theorem.

Theorem 1. *If N is a positive even integer, and i is co-prime with $N + 2i + 1$, then we have an estimation*

$$x_{N,i}(n) \geqslant y_{N+2i+1,i}(n), \tag{3}$$

for general positive integer ν, $i < \frac{\nu}{2}$ and co-prime with ν, we have an estimation

$$y_{\nu,1}(n) \geqslant y_{\nu,i}(n). \tag{4}$$

We mainly study for $N \geqslant 4$ because the results are known for $N < 4$.

2. Vertex operators

As known, affine Lie algebra

$$\widetilde{sl_2(\mathbb{C})} = sl_2(\mathbb{C}) \otimes \mathbb{C}\left[t^{-1}, t\right] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

has an infinite dimensional Heisenberg Lie subalgebra

$$\widetilde{H} = \bigoplus_{n \in \mathbb{Z} \setminus \{0\}} \mathbb{C}h(n) \oplus \mathbb{C}c, \tag{5}$$