Journal of Number Theory 132 (2012) 2673–2699

Contents lists available at SciVerse ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On sums of Apéry polynomials and related congruences ☆

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ARTICLE INFO

Article history:
Received 23 July 2011
Revised 29 May 2012
Accepted 29 May 2012
Available online 21 July 2012
Communicated by David Goss

MSC:
primary 11A07, 11B65
secondary 05A10, 11B68, 11E25

Keywords:
Apéry numbers and Apéry polynomials
Bernoulli numbers
Binomial coefficients
Congruences

ABSTRACT

The Apéry polynomials are given by

$$A_n(x)=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2x^k\quad(n=0,1,2,\ldots).$$

(Those $A_n=A_n(1)$ are Apéry numbers.) Let p be an odd prime. We show that

$$\sum_{k=0}^{p-1}(-1)^kA_k(x)\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k}x^k\pmod{p^2},$$

and that

$$\sum_{k=0}^{p-1}A_k(x)\equiv\left(\frac{x}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{4k}{k,k,k,k}}{(256x)^k}\pmod{p}$$

for any p-adic integer $x\not\equiv 0\pmod p$. This enables us to determine explicitly $\sum_{k=0}^{p-1}(\pm1)^kA_k \bmod p$, and $\sum_{k=0}^{p-1}(-1)^kA_k \bmod p^2$ in the case $p\equiv 2\pmod 3$. Another consequence states that

$$\sum_{k=0}^{p-1}(-1)^kA_k(-2)\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p=x^2+4y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } p\equiv 3\pmod 4.\end{cases}$$

☆ Supported by the National Natural Science Foundation (grant 11171140) of China and the PAPD of Jiangsu Higher Education Institutions.

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0022-314X/$ – see front matter
http://dx.doi.org/10.1016/j.jnt.2012.05.014

We also prove that for any prime $p>3$ we have

$$\sum_{k=0}^{p-1}(2k+1)A_k\equiv p+\frac{7}{6}p^4B_{p-3}\pmod{p^5}$$

where $B_0, B_1, B_2, \ldots$ are Bernoulli numbers.

1. Introduction

The well-known Apéry numbers given by

$$A_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2=\sum_{k=0}^{n}\binom{n+k}{2k}^2\binom{2k}{k}^2\quad(n\in\mathbb{N}=\{0,1,2,\ldots\})$$

play a central role in Apéry's proof of the irrationality of $\zeta(3)=\sum_{n=1}^{\infty}1/n^3$ (see Apéry [Ap] and van der Poorten [Po]). They also have close connections to modular forms (cf. Ono [O, pp. 198–203]). The Dedekind eta function in the theory of modular forms is defined by

$$\eta(\tau)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\quad\text{with } q=e^{2\pi i\tau},$$

where $\tau\in\mathbb{H}=\{z\in\mathbb{C}:\ \mathrm{Im}(z)>0\}$ and hence $|q|<1$. In 1987 Beukers [B] conjectured that

$$A_{(p-1)/2}\equiv a(p)\pmod{p^2}\quad\text{for any prime } p>3,$$

where $a(n)$ $(n=1,2,3,\ldots)$ are given by

$$\eta^4(2\tau)\eta^4(4\tau)=q\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{4n})^4=\sum_{n=1}^{\infty}a(n)q^n.$$

This was finally confirmed by Ahlgren and Ono [AO] in 2000.

We define Apéry polynomials by

$$A_n(x)=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2x^k=\sum_{k=0}^{n}\binom{n+k}{2k}^2\binom{2k}{k}^2x^k\quad(n\in\mathbb{N}).\tag{1.1}$$

Clearly $A_n(1)=A_n$. Motivated by the Apéry polynomials, we also introduce a new kind of polynomials:

$$W_n(x):=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n-k}{k}^2x^k=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}^2\binom{2k}{k}^2x^k\quad(n\in\mathbb{N}).\tag{1.2}$$