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Zeros of the Riemann zeta function on the critical line

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ABSTRACT

We introduce a new mollifier and apply the method of Levinson and Conrey to prove that at least 41.28% of the zeros of the Riemann zeta function are on the critical line. The method may also be used to improve other results on zeros relate to the Riemann zeta function, as well as conditional results on prime gaps.

1. Introduction

Let $\zeta(s)$ denote the Riemann zeta function, where $s = \sigma + it$. It is defined for $\sigma > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p} \left(1 - p^{-s}\right)^{-1},$$

where p runs over the prime numbers, and has a meromorphic continuation to the whole complex plane with its only pole, a simple pole at $s = 1$. It satisfies the functional equation (see [20])

$$\xi(s) = \xi(1 - s), \tag{1.1}$$

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where the entire function $\xi(s)$ is defined by

$$\xi(s) = H(s)\zeta(s) \tag{1.2}$$

with

$$H(s) = \frac{s(s-1)}{2}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right). \tag{1.3}$$

Let $N(T)$ denote the number of zeros of $\zeta(s)$, $s = \sigma + it$, in the rectangle $0 < \sigma < 1$, $0 < t \leqslant T$, each zeros is counted with multiplicity, von Mangoldt proved that (see [24])

$$N(T) = \frac{T}{2\pi}\left(\log\frac{T}{2\pi} - 1\right) + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right),$$
$$S(T) = \frac{1}{\pi}\arg\zeta\left(\frac{1}{2} + iT\right) = O(\log T), \quad \text{as } T \to \infty.$$

Let $N_0(T)$ be the number of zeros of $\zeta(\frac{1}{2} + it)$ on $0 < t \leqslant T$, each zeros is counted with multiplicity, $N_{0s}(T)$ be the number of simple zeros of $\zeta(\frac{1}{2} + it)$ on $0 < t \leqslant T$. The Riemann Hypothesis says that $N_0(T) = N(T)$, the Simple Zeros Conjecture combined with the Riemann Hypothesis says that $N_{0s}(T) = N_0(T) = N(T)$.

It was proved for the first time by Hardy [11] in 1914 that $\zeta(s)$ has infinitely many zeros on the critical line $\sigma = \frac{1}{2}$, thus

$$N_0(T) \to \infty \quad \text{as } T \to \infty.$$

Hardy's qualitative result was given a quantitative form

$$N_0(T) \geqslant AT$$

for some $A > 0$ and T large enough, by Hardy and Littlewood [12] in 1921, and later, with an explicit value of A, the same result was obtained by Siegel [23] in 1932 with a rather different method.

Let

$$\kappa = \liminf_{T\to\infty}\frac{N_0(T)}{N(T)}, \tag{1.4}$$

$$\kappa^* = \liminf_{T\to\infty}\frac{N_{0s}(T)}{N(T)}. \tag{1.5}$$

In 1942, Selberg [21] proved that there is an effectively computable positive constant A such that

$$\kappa \geqslant A.$$

Selberg's proof involved combing a 'mollifier' to compensate for irregularities in the size of $|\zeta(s)|$ and the method of Hardy and Littlewood.

In 1974, Levinson [15] combined Siegel's idea and Selberg's idea and proved that

$$\kappa \geqslant 0.3420.$$

The Levinson method involve the following main issues: