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π and the hypergeometric functions of complex argument

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ABSTRACT

Text. In this article we derive some new identities concerning π, algebraic radicals and some special occurrences of the Gauss hypergeometric function ${}_2F_1$ in the analytic continuation. All of them have been derived by tackling some elliptic or hyperelliptic known integral, and looking for another representation of it by means of hypergeometric functions like those of Gauss, Appell or Lauricella. In any case we have focused on integrand functions having at least one couple of complex-conjugate roots. Founding upon a special hyperelliptic reduction formula due to Hermite (1876) [6], π is obtained as a ratio of a complete elliptic integral and the four-variable Lauricella function. Furthermore, starting with a certain binomial integral, we succeed in providing $\sqrt{2}/3$ as a ratio of a linear combination of complete elliptic integrals of the first and second kinds to the Appell hypergeometric function of two complex-conjugate arguments. Each of the formulae we found theoretically has been satisfactorily tested by means of Mathematica®.

Video. For a video summary of this paper, please click here or visit http://www.youtube.com/watch?v=rQqtVtAf-RQ.

1. Introduction

In this article some new identities concerning π and other relevant numbers are obtained following a methodology like that which appeared in our previous paper [9], which the reader is referred to, also for a review on the recent literature on π formulae. Hereinafter we will focus on some integrals not considered in [9], namely those with complex-conjugate roots, i.e. like

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$$\int \frac{p(x)}{\sqrt{q(x)}}\,\mathrm{d}x$$

being $q(x)$ a third/fourth degree real coefficients polynomial with almost one couple of complex-conjugate roots and $p(x)$ with degree 0 or 1.

In the second section, by means of elliptic integrals, whose radicands always have complex-conjugate roots, evaluated by [3], we obtain further identities of elliptic-hypergeometric nature, not only to π but also to algebraic radicals like $\sqrt{2}$, $\sqrt[4]{3}$. Our most prominent outcomes appear to be where the hypergeometric functions ${}_2\mathrm{F}_1$ (Gauss), F_1 (Appell) and $\mathrm{F}_D^{(n)}$ (Lauricella) are theoretically found inside the unit disk, or in their analytic continuation. In the third section, starting from the integral where $2\alpha - \beta > 1$:

$$\int_0^\infty \frac{t^\beta}{(1+t^2)^\alpha}\,\mathrm{d}t$$

we will establish some determinations, completely new as we deem, on the Gauss hypergeometric function ${}_2\mathrm{F}_1$ with argument 2, without making use of any formulae on analytic continuation.

In the fourth section, a hyperelliptic subject takes place. A special hyperelliptic integral can in fact be reduced to an elliptic one, by the variable transformation:

$$y = \frac{2(z^3 - b^3)}{3(z^2 - a^2)}$$

leading to the reduction (Hermite 1876 [6]):

$$\int_{z_1}^\infty \frac{z}{\sqrt{(z^2 - a^2)(4z^3 - 3a^2 z - b^3)}}\,\mathrm{d}z = \frac{1}{\sqrt{6}} \int_{-2z_1}^\infty \frac{\mathrm{d}y}{\sqrt{y^3 - 3a^2 y + 2b^3}}. \tag{1}$$

Founding upon (1), if $b > a > 0$, defining $q_1(z) = 4z^3 - 3a^2 z - b^3$, $q_2(y) = y^3 - 3a^2 y + 2b^3$ we have two polynomials that have only one real root: moreover, there exists $z_1 > 0$ such that $q_1(z_1) = 0$, $q_2(-2z_1) = 0$. Notice that each root (real or complex) of $q_2(y)$ can be obtained by multiplying the $q_1(z)$ roots by -2. Next we will use the double approach (hypergeometric and elliptic) to compute some integrals which we have not dealt with before [9], where all the roots of $q(x)$ are real and the interval of integration is bounded. In [9], integrals with real roots had been carried out over bounded intervals as per the famous hypergeometric-elliptic Jacobi (1832) reduction shown in [7]:

$$\int_0^1 \frac{(\sqrt{ab} + z)\,\mathrm{d}z}{\sqrt{z(z-1)(z-ab)(z-a)(z-b)}} = \frac{1}{\sqrt{(1-a)(1-b)}} \int_0^1 \frac{\mathrm{d}y}{\sqrt{y(1-y)(1-c_{ab}y)}}$$

where

$$c_{ab} = -\frac{(\sqrt{a} - \sqrt{b})^2}{(1-a)(1-b)}.$$

We will see how the integral on the left-hand side of (1) can be evaluated via the same methods described in [9] through the Lauricella hypergeometric function $\mathrm{F}_D^{(4)}$, some argument of which shall be necessarily complex. As a consequence, a new π identity will follow once the (1) integral has been expressed as a complete first kind elliptic integral, formula 241.00, p. 88 of [3]. In this case the