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Moments of combinatorial and Catalan numbers [☆]

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ABSTRACT

In this paper we obtain the moments $\{\Phi_m\}_{m\geqslant 0}$ defined by

$$\Phi_m(n) := \sum_{p=1}^{n+1} (2p-1)^m \binom{2n+1}{n+1-p}^2, \qquad n\in\mathbb{N},\ m\in\mathbb{N}_0 := \mathbb{N}\cup\{0\},$$

where $\binom{m}{n}$ is the usual combinatorial number. We also provide the moments in the Catalan triangle whose (n,p) entry is defined by

$$A_{n,p} := \frac{2p-1}{2n+1}\binom{2n+1}{n+1-p}, \qquad n,p\in\mathbb{N},\ p\leqslant n+1,$$

and, in particular, new identities involving the well-known Catalan numbers.

0. Introduction

Although there exist several triangles known as the "Catalan triangle", the following one is one of the most-standing form

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$k \setminus m$	0	1	2	3	4	5	6	...
0	1							
1	1	1						
2	1	2	2					
3	1	3	5	5				
4	1	4	9	14	14			
5	1	5	14	28	42	42		
6	1	6	20	48	90	132	132	
...	...	...	...	...	...	...	...	

see for example [10]. Each entry $C_{k,m}$ is defined by

$$C_{k,m} := \frac{(k+m)!(k-m+1)}{m!(k+1)!}, \quad 0 \leqslant m \leqslant k.$$

Notice that $C_{k,k}$ is the well-known Catalan number C_k, given by the formula

$$C_k = \frac{1}{k+1}\binom{2k}{k}, \quad k \geqslant 1.$$

The Catalan numbers may be defined recursively by $C_0 = 1$ and $C_k = \sum_{i=0}^{k-1} C_i C_{k-1-i}$ for $k \geqslant 1$. These numbers appear in a wide range of problems, see [11]. For instance, the Catalan number C_k counts the number of ways to triangulate a regular polygon with $k+2$ sides; or, let $2k$ people seat around a circular table, the Catalan number C_k gives the number of ways that all of them are simultaneously shaking hands with another person at the table in such a way that none of the arms cross each other.

In the Catalan triangle, we now consider numbers $C_{k,m}$ in the same diagonal such that $k+m$ is odd. We write $k+m = 2n-1$ and $p = n-m$ to get Shapiro's triangle introduced in [8],

$n \setminus p$	1	2	3	4	5	6	...
1	1						
2	2	1					
3	5	4	1				
4	14	14	6	1			
5	42	48	27	8	1		
6	132	165	110	44	10	1	
...	...	...	...	...	...	...	...

(1)

whose entries are given by

$$B_{n,p} := \frac{p}{n}\binom{2n}{n-p}, \quad n, p \in \mathbb{N}, \ p \leqslant n.$$

On the other hand, when $k+m = 2n$ and $p = n-m+1$, we recover the following triangle