Journal of Number Theory 129 (2009) 1669–1677

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On certain exponential sums over primes

H. Maier [a], A. Sankaranarayanan [b,*,1]

[a] Institute for Number Theory and Probability Theory, University of Ulm, D-89069 Ulm, Germany
[b] School of Mathematics, TIFR, Homi Bhabha Road, Mumbai 400 005, Maharashtra, India

ARTICLE INFO

Article history:
Received 9 July 2008
Revised 7 January 2009
Available online 20 March 2009
Communicated by K. Soundararajan

MSC:
primary 11L20
secondary 11L07, 11L15

Keywords:
Exponential sums
von Mangoldt function
Vaughan's identity
Difference operator
Hölder's inequality

ABSTRACT

Let $f(x)$ be a real valued polynomial in x of degree $k \geqslant 4$ with leading coefficient α. In this paper, we prove a non-trivial upper bound for the quantity

$$\left|\sum_{p \leqslant N} (\log p) e\big(f(p)\big)\right|$$

whenever the leading coefficient α of $f(x)$ is of type 1.

1. Introduction

The estimations of exponential and related sums are of great importance in number theory. A more general problem is to estimate an upper bound for the quantity

$$\left|\sum_{n \leqslant N} a_n e\big(f(n)\big)\right|, \tag{1.1}$$

* Corresponding author.
E-mail addresses: helmut.maier@uni-ulm.de (H. Maier), ayyadurai.sankaranarayanan@uni-ulm.de, sank@math.tifr.res.in (A. Sankaranarayanan).
[1] Current Address: Institute for Number Theory and Probability Theory, University of Ulm, D-89069 Ulm, Germany.

doi:10.1016/j.jnt.2009.01.018

where a_n's are certain complex numbers and $f(n)$ is a nice function. When $a_n = \Lambda(n)$ (the von Mangoldt function), the estimation of the sum S in question essentially turns out to be estimating a certain exponential sum over primes, more precisely,

$$\left|\sum_{n\leqslant N} \Lambda(n)e\big(f(n)\big)\right| = \sum_{p\leqslant N} (\log p)e\big(f(p)\big) + O\big(N^{\frac{1}{2}}\big). \tag{1.2}$$

Throughout the paper, $f(x)$ is a real valued polynomial of degree k ($\geqslant 2$) with the leading coefficient α. There are several interesting results available in the literature. For example, G. Harman proved the following theorem (see Theorem 1) in [4].

Theorem A. *Suppose $\epsilon > 0$ is given. Let $\gamma_1(k) = 4^{1-k}$. Suppose that there are integers a, q such that*

$$|q\alpha - a| < q^{-1} \quad \text{with } (a,q) = 1.$$

Then we have

$$\sum_{p\leqslant N} (\log p)e\big(f(p)\big) \ll N^{1+\epsilon}\left(\frac{1}{q} + \frac{1}{N^{\frac{1}{2}}} + \frac{q}{N^k}\right)^{\gamma_1(k)}.$$

For an application of Theorem A with large q, we refer to [1]. A. Ghosh considered some special cases of the above sum when $f(p) = \alpha p^2$ or αp^3 (see [3]). It should be mentioned here that K. Kawada and T.D. Wooley have studied estimations of sums of the kind $\sum_{P\leqslant p<2P} e(\alpha p^k)$ with some restrictions on α in connection with the Waring–Goldbach problem for fourth and fifth powers (see [7] and also [6]). The above estimates are in general good whenever the degree of $f(x)$ is small. On the other hand, if k is large, Vinogradov's (see [20]) result shows that in place of $\gamma_1(k)$ in Theorem A, we can have $(25k^2(2+\log k))^{-1}$. If f is a monomial and α is rational, then Theorem 2 of [14] shows that, Theorem A can be substantially improved to

$$\sum_{p\leqslant N} (\log p)e\left(\frac{ap^k}{q}\right) \ll (\log N)^{7/2} q^{\epsilon}\big(N^{\frac{1}{2}}q^{\frac{1}{2}} + Nq^{-\frac{1}{2}} + N^{\frac{3}{4}}q^{\frac{1}{8}}\big) \tag{1.3}$$

(see also the related works [2,5,15,16,18,19]).

Since our Main Theorem below depends on the type of α, let us recall this notion (see p. 121 of [9] for more details). Let ψ be a non-decreasing positive function that is defined at least for all positive integers. The irrational number α is said to be of type $< \psi$ if $q\|q\alpha\| \geqslant \frac{1}{\psi(q)}$ holds for all positive integers q. If ψ is a constant function, then an irrational α of type $< \psi$ is also called of constant type. Let η_1 be a positive real number or infinity. The irrational number α is said to be of type η_1 if η_1 is the supremum of all δ_1 for which

$$\liminf_{q\to\infty} q^{\delta_1}\|q\alpha\| = 0,$$

where q runs through the positive integers. The relationship between these two definitions is that an irrational number α is of type η_1 if and only if for every $\tau > \eta_1$ there is a constant $c = c(\tau,\alpha)$ such that α is of type $< \psi$ where $\psi(q) = cq^{\tau-1}$. It is well known that almost all numbers are of type 1. From Roth's theorem, we note that all algebraic irrationalities α satisfy