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Congruences involving Bernoulli and Euler numbers

Zhi-Hong Sun

Department of Mathematics, Huaiyin Teachers College, Huaian, Jiangsu 223001, PR China

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Abstract

Let $[x]$ be the integral part of x. Let $p>5$ be a prime. In the paper we mainly determine $\sum_{x=1}^{[p/4]}\frac{1}{x^k} \pmod{p^2}$, $\binom{p-1}{[p/4]} \pmod{p^3}$, $\sum_{k=1}^{p-1}\frac{2^k}{k} \pmod{p^3}$ and $\sum_{k=1}^{p-1}\frac{2^k}{k^2} \pmod{p^2}$ in terms of Euler and Bernoulli numbers. For example, we have

$$\sum_{x=1}^{[p/4]}\frac{1}{x^2}\equiv(-1)^{\frac{p-1}{2}}(8E_{p-3}-4E_{2p-4})+\frac{14}{3}pB_{p-3} \pmod{p^2},$$

where E_n is the nth Euler number and B_n is the nth Bernoulli number.

MSC: primary 11B68; secondary 11A07

Keywords: Congruence; Bernoulli number; Bernoulli polynomial; Euler number

1. Introduction

The Bernoulli numbers $\{B_n\}$ and Bernoulli polynomials $\{B_n(x)\}$ are defined by

$$B_0=1,\quad \sum_{k=0}^{n-1}\binom{n}{k}B_k=0 \quad (n\geqslant 2) \quad\text{and}\quad B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k} \quad (n\geqslant 0).$$

E-mail address: hyzhsun@public.hy.js.cn.
URL: http://www.hytc.edu.cn/xsjl/szh.

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The Euler numbers $\{E_n\}$ and Euler polynomials $\{E_n(x)\}$ are defined by

$$\frac{2e^t}{e^{2t}+1}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!}\left(|t|<\frac{\pi}{2}\right) \quad \text{and} \quad \frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!} \quad \bigl(|t|<\pi\bigr),$$

which are equivalent to (see [MOS])

$$E_0=1, \quad E_{2n-1}=0, \quad \sum_{r=0}^{n}\binom{2n}{2r}E_{2r}=0 \quad (n\geqslant 1)$$

and

$$E_n(x)=\frac{1}{2^n}\sum_{r=0}^{n}\binom{n}{r}(2x-1)^{n-r}E_r.$$

Let $[x]$ be the integral part of x. For a given prime p let $\mathbb{Z}_p$ denote the set of rational p-integers (those rational numbers whose denominator is not divisible by p). For $a\in\mathbb{Z}_p$ with $a\not\equiv 0 \pmod p$, as usual we define the Fermat quotient $q_p(a)=(a^{p-1}-1)/p$. In the paper we establish some congruences involving Bernoulli and Euler numbers. In particular, in $\mathbb{Z}_p$ we have

$$\sum_{\frac{p}{4}<k<\frac{p}{2}}\frac{1}{k}\equiv q_p(2)-p\left(\frac{1}{2}q_p(2)^2+(-1)^{\frac{p-1}{2}}(E_{2p-4}-2E_{p-3})\right)+\frac{1}{3}p^2q_p(2)^3 \pmod{p^3},$$

$$(-1)^{[\frac{p}{4}]}\binom{p-1}{[\frac{p}{4}]}\equiv 1+3pq_p(2)+p^2\bigl(3q_p(2)^2-(-1)^{\frac{p-1}{2}}E_{p-3}\bigr) \pmod{p^3},$$

$$\sum_{\substack{1\leqslant k<p\\ 4\mid k+p}}\frac{1}{k}\equiv\frac{1}{4}q_p(2)-\frac{1}{8}pq_p(2)^2+\frac{1}{12}p^2q_p(2)^3-\frac{7}{192}p^2B_{p-3} \pmod{p^3},$$

$$\sum_{k=1}^{p-1}\frac{2^k}{k}\equiv -2q_p(2)-\frac{7}{12}p^2B_{p-3} \pmod{p^3},$$

$$\sum_{k=1}^{p-1}\frac{2^k}{k^2}\equiv -q_p(2)^2+p\left(\frac{2}{3}q_p(2)^3+\frac{7}{6}B_{p-3}\right) \pmod{p^2},$$

where p is a prime greater than 5.

In addition to the above notation, we also use throughout this paper the following notation: $\mathbb{Z}$—the set of integers, $\mathbb{N}$—the set of positive integers, $\{x\}$—the fractional part of x, $\varphi(n)$—Euler's totient function.

2. Basic lemmas

We begin with a useful identity involving Bernoulli polynomials.