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On recurrences for sums of powers of binomial coefficients

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Abstract

The recurrence for sums of powers of binomial coefficients is considered and a lower bound for the minimal length of the recurrence is obtained by using the properties of congruence.
Video abstract: For a video summary of this paper, please visit http://www.youtube.com/watch?v=jwy6B4aYR-Q.

Keywords: Binomial coefficients; Congruence; Recurrence

1. Introduction

For any $r \in N = \{1, 2, \ldots\}$ we consider the sums

$$a_n^{(r)} = S_r(n) = \sum_{k=0}^{n} \binom{n}{k}^r, \quad n \geqslant 0. \tag{1}$$

These sums have been studied by many authors. Apart from the trivial recurrences for $S_1(n) = 2^n$ and $S_2(n) = \binom{2n}{n}$, Calkin [1] claims (citing Wilf) that for $3 \leqslant a \leqslant 9$, there is no closed form for $s_r(n)$. J. Franel [3,4] was the first to obtain recurrences for $S_3(n)$ and $S_4(n)$, namely

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$$P_0(n)S_r(n+1) + P_1(n)S_r(n) + P_2(n)S_r(n-1) = 0, \quad n \geqslant 0, \tag{2}$$

where for $r = 3$:

$$P_0(n) = (n+1)^2, \qquad P_1(n) = -\bigl(7n^2 + 7n + 2\bigr), \qquad P_2(n) = -8n^2$$

and for $r = 4$:

$$P_0(n) = (n+1)^3, \qquad P_1(n) = -2(2n+1)\bigl(3n^2 + 3n + 1\bigr),$$
$$P_2(n) = -4n(4n+1)(4n-1).$$

Franel also conjectured that $S_r(n)$ would satisfy similar recurrences for each $r \in N$, and more precisely of length $[\frac{1}{2}(r+3)]$ and of polynomial degree $\leqslant r-1$. The existence of such recurrences was first proved by R.P. Stanley [7], however without any bounds on the lengths and degrees. For $r = 5, 6$ M.A. Perlstadt [6] found recurrences of length 4, namely

$$P_0(n)S_r(n+1) + P_1(n)S_r(n) + P_2(n)S_r(n-1) + P_3(n)S_r(n-2) = 0, \quad n \geqslant 0,$$

where for $r = 5$:

$$\begin{aligned}
P_0(n) &= (n+1)^4\bigl(55n^2 - 77n + 28\bigr),\\
P_1(n) &= -1155n^6 - 693n^5 + 732n^4 + 715n^3 - 45n^2 - 210n - 56,\\
P_2(n) &= -19415n^6 + 27181n^5 - 7453n^4 - 3289n^3 + 956n^2 + 276n - 96,\\
P_3(n) &= 32(n-1)^4\bigl(55n^2 + 33n + 6\bigr)
\end{aligned}$$

and for $r = 6$:

$$\begin{aligned}
P_0(n) &= n(n+1)^5\bigl(91n^3 - 182n^2 + 126n - 30\bigr),\\
P_1(n) &= -n\bigl(3458n^8 + 1729n^7 - 2947n^6 - 2295n^5 + 901n^4 + 1190n^3 + 52n^2 - 228n - 60\bigr),\\
P_2(n) &= -153881n^9 + 307762n^8 - 185311n^7 - 2960n^6 + 31631n^5 + 88n^4 - 5239n^3\\
&\qquad + 610n^2 + 440n - 100,\\
P_3(n) &= 24(n-1)^3(2n-1)(6n-7)(6n-5)\bigl(91n^3 + 91n^2 + 35n + 5\bigr).
\end{aligned}$$

It was proved by T.W. Cusick [2] that $S_r(n)$ as predicted by Franel actually satisfies a polynomial recurrence of length $[\frac{1}{2}(r+3)]$ for all $r \geqslant 1$, however, as pointed out by M. Stoll [8], there is a gap in Cusick's argument. On the other hand Stoll proved Franel's by extending Stanly's approach.

For $7 \leqslant r \leqslant 10$ R.J. McIntosh [5] used Cusick's method to compute such recurrences.

For $S_r(n)$ no lower bounds ($r > 2$) for the lengths of recurrences have been known (even for $r = 3, 4$).

In some similar cases it is possible to get lower bounds from the asymptotics of the sequences (e.g. for Apéry's sequence). However for the sequence $S_r(n)$ one has the asymptotic formula