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Maximal unramified 3-extensions of imaginary quadratic fields and $SL_2(\mathbb{Z}_3)$

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Abstract

The structure of the Galois group of the maximal unramified p-extension of an imaginary quadratic field is restricted in various ways. In this paper we construct a family of finite 3-groups satisfying these restrictions. We prove several results about this family and characterize them as finite extensions of certain quotients of a Sylow pro-3 subgroup of $SL_2(\mathbb{Z}_3)$. We verify that the first group in the family does indeed arise as such a Galois group and provide a small amount of evidence that this may hold for the other members. If this was the case then it would imply that there is no upper bound on the possible lengths of a finite p-class tower.

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1. Maximal unramified p-extensions and Schur-σ groups

Let k be an imaginary quadratic number field and p be a prime. The p-class tower of k is the sequence of fields

$$k = k_1 \subseteq k_2 \subseteq k_3 \subseteq \cdots$$

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where k_{n+1} is the maximal unramified abelian p-extension of k_n. By Galois theory the fields k_n correspond to the subgroups in the derived series of $G = \mathrm{Gal}(k^{nr,p}/k)$ where $k^{nr,p} = \bigcup_{n \geqslant 1} k_n$ is the maximal unramified p-extension of k. If we let $Cl_p(F)$ denote the p-Sylow subgroup of the class group of a number field F then by class field theory $\mathrm{Gal}(k_{n+1}/k_n) \cong Cl_p(k_n)$ for $n \geqslant 1$. In particular $G/[G,G] \cong \mathrm{Gal}(k_2/k_1) \cong Cl_p(k)$ and so is finite.

Now assume also that $p \neq 2$. In [11] the notion of a Schur-σ group is introduced. It encapsulates various properties that the Galois group G is known to satisfy in this case. These are:

1. The generator rank and relation rank of G (as a pro-p group) are equal;
2. $G/[G,G]$ is finite;
3. There exists an automorphism σ of order 2 on G which induces the inverse automorphism $a \mapsto a^{-1}$ on $G/[G,G]$.

Several structural results are proved there about the presentations of such groups. One consequence of their work is that if $d(G) \geqslant 3$ then the extension $k^{nr,p}/k$ is infinite. It follows that all such extensions which are finite and non-abelian must have $d(G) = 2$.

In general it is exceedingly difficult to compute the Galois group G. For those examples in which the group is known to be finite the length of the derived series is usually small. Indeed to date the largest length observed is 3 and in all these examples $p = 2$, see [5]. In the next section we will define a family of finite Schur-σ groups with $p = 3$ and then show that the derived length for groups in this family is unbounded. In the last section we show that the first group in the family is isomorphic to $\mathrm{Gal}(k^{nr,p}/k)$ for several different choices of k.

2. A family of Schur σ-groups of unbounded derived length

Let F be the free pro-3 group on two generators x and y. Let G_n be the Schur-σ group defined by the pro-3 presentation

$$G_n = \langle x, y \mid r_n^{-1}\sigma(r_n), t^{-1}\sigma(t) \rangle$$

where $r_n = x^3 y^{-3^n}$, $t = yxyx^{-1}y$ and $\sigma : F \to F$ is the automorphism defined by $x \mapsto x^{-1}$ and $y \mapsto y^{-1}$. We will prove the following result.

Theorem 2.1. *For $n \geqslant 1$ the following hold*:

(i) *G_n is a finite 3-group of order 3^{3n+2}*;
(ii) *G_n is nilpotent of class $2n+1$*;
(iii) *G_n has derived length $\lfloor \log_2(3n+3) \rfloor$*.

The remainder of this section is devoted to the proof. We first define some auxiliary groups which are easier to study than G_n. Let H_n be given by the pro-3 presentation

$$H_n = \langle x, y \mid x^3, y^{3^n}, t^{-1}\sigma(t) \rangle,$$

and let H be given by the pro-3 presentation

$$H = \langle x, y \mid x^3, t^{-1}\sigma(t) \rangle.$$