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# Normality of numbers generated by the values of entire functions ☆

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**Abstract**

We show that the number generated by the $q$-ary integer part of an entire function of logarithmic order, where the function is evaluated over the natural numbers and the primes, respectively, is normal in base $q$. This is an extension of related results for polynomials over the real numbers established by Nakai and Shiokawa.


## 1. Introduction

Let $q \geqslant 2$ be a fixed integer and $\theta = 0.a_1a_2\dots$ be the $q$-ary expansion of a real number $\theta$ with $0 < \theta < 1$. We write $d_1 \dots d_l \in \{0, 1, \dots, q-1\}^l$ for a block of $l$ digits in the $q$-ary expansion. By $\mathcal{N}(\theta; d_1 \dots d_l; N)$ we denote the number of occurrences of the block $d_1 \dots d_l$ in the first $N$ digits of the $q$-ary expansion of $\theta$. We call $\theta$ *normal to the base q* if for every fixed $l \geqslant 1$

$$\mathcal{R}_N(\theta) = \mathcal{R}_{N,l}(\theta) = \sup_{d_1\dots d_l} \left| \frac{1}{N}\mathcal{N}(\theta; d_1 \dots d_l; N) - \frac{1}{q^l} \right| = o(1)$$

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as $N \to \infty$, where the supremum is taken over all blocks $d_1 \ldots d_l \in \{0, 1, \ldots, q-1\}^l$.

We want to look at numbers whose digits are generated by the integer part of entire functions. Let $f$ be any function and $[f(n)]_q$ denote the base $q$ expansion of the integer part of $f(n)$, then define

$$\theta_q = \theta_q(f) = 0.\big[f(1)\big]_q\big[f(2)\big]_q\big[f(3)\big]_q\big[f(4)\big]_q\big[f(5)\big]_q\big[f(6)\big]_q\cdots,$$
$$\tau_q = \tau_q(f) = 0.\big[f(2)\big]_q\big[f(3)\big]_q\big[f(5)\big]_q\big[f(7)\big]_q\big[f(11)\big]_q\big[f(13)\big]_q\cdots, \tag{1.1}$$

where the sequences of the arguments run through the positive integers and the primes, respectively.

In this paper we consider the construction of normal numbers in base $q$ as concatenation of $q$-ary integer parts of certain functions. The first result on that topic was achieved by Champernowne [2], who was able to show that

$$0.1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,12\,13\,14\,15\,16\,17\,18\,19\,20\ldots$$

is normal in base 10. This construction can be easily generalised to any integer base $q$. Copeland and Erdös [4] were able to show that

$$0.2\,3\,5\,7\,11\,13\,17\,19\,23\,29\,31\,37\,41\,43\,47\,53\,59\,61\,67\ldots$$

is normal in base 10. These examples correspond to the choice $f(x) = x$ in (1.1). Davenport and Erdös [5] considered the case where $f(x)$ is a polynomial whose values at $x = 1, 2, \ldots$ are always integers and showed that in this case the numbers $\theta_q(f)$ and $\tau_q(f)$ are normal. For $f(x)$ a polynomial with *rational* coefficients Schiffer [10] was able to show that $\mathcal{R}_N(\theta_q(f)) = \mathcal{O}(1/\log N)$. Nakai and Shiokawa [8] extended his results and showed that $\mathcal{R}_N(\tau_q(f)) = \mathcal{O}(1/\log N)$. In the case of *real* coefficients Nakai and Shiokawa [7] proved the same estimate for $\mathcal{R}_N(\theta_q(f))$. In this paper we want to discuss the case where $f(x)$ is a *transcendental entire function* (i.e., an entire function that is not a polynomial) of small *logarithmic order*. Recall that we say an increasing function $S(r)$ has *logarithmic order* $\lambda$ if

$$\limsup_{r\to\infty} \frac{\log S(r)}{\log\log r} = \lambda. \tag{1.2}$$

We define the *maximum modulus* of an entire function $f$ to be

$$M(r, f) := \max_{|x|\leqslant r} \big|f(x)\big|. \tag{1.3}$$

If $f$ is an entire function and $\log M(r, f)$ has logarithmic order $\lambda$, then we call $f$ an *entire function of logarithmic order* $\lambda$.

To achieve our results we combine the following ingredients:

- The first part of the proofs concerns the estimation for the number of solutions of the equation $f(x) = a$ where $a \in \mathbb{C}$ (cf. [3], [11, Section 8.21]) for entire functions of zero order.
- Following the methods of Nakai and Shiokawa [7,8] we reformulate the problem in an estimation of exponential sums.