Journal of Number Theory 129 (2009) 729–738

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

A note on a refined version of Anderson–Brownawell–Papanikolas criterion

Chieh-Yu Chang [a,b,*,1]

[a] National Center for Theoretical Sciences, Department of Mathematics, National Tsing Hua University, Hsinchu City 300, Taiwan, ROC
[b] Department of Mathematics, National Central University, Chung-Li 32054, Taiwan, ROC

ARTICLE INFO

Article history:
Received 12 March 2008
Available online 31 December 2008
Communicated by Dinesh S. Thakur

MSC:
primary 11J93, 11G09

Keywords:
ABP criterion
Linear independence
Algebraic independence

ABSTRACT

We give a refinement of the linear independence criterion over function fields developed by Anderson, Brownawell and Papanikolas [Greg W. Anderson, W. Dale Brownawell, Matthew A. Papanikolas, Determination of the algebraic relations among special Γ-values in positive characteristic, Ann. of Math. 160 (2004) 237–313]. As a consequence, a function field analogue of the Siegel–Shidlovskii theorem is derived.

1. Introduction

Let $\mathbb{F}_q$ be the finite field of q elements with characteristic p. Let $A := \mathbb{F}_q[\theta]$ be the polynomial ring in variable θ over $\mathbb{F}_q$, with fraction field $k := \mathbb{F}_q(\theta)$. Define an absolute value $|\cdot|_\infty$ at the infinite place of k so that $|\theta|_\infty = q$. Let $k_\infty := \mathbb{F}_q((\frac{1}{\theta}))$ be the ∞-adic completion of k, let $\overline{k_\infty}$ be a fixed algebraic closure of k_∞, let $\mathbb{C}_\infty$ be the ∞-adic completion of $\overline{k_\infty}$, let $\bar{k}$ be the algebraic closure of k in $\mathbb{C}_\infty$ and let $\overline{\mathbb{F}_q}$ be the algebraic closure of $\mathbb{F}_q$ in $\bar{k}$.

Let t be an independent variable of θ. Let $\mathbb{T}$ be the Tate algebra of power series in $\mathbb{C}_\infty[[t]]$ that are convergent on the closed unit disk in $\mathbb{C}_\infty$, and let $\mathbb{L} \subseteq \mathbb{C}_\infty((t))$ be the fraction field of $\mathbb{T}$. Let $\mathbb{E}$ be the subring of $\mathbb{T}$ consisting of power series that are everywhere convergent and whose coefficients

* Address for correspondence: Mathematics Division, National Center for Theoretical Sciences, National Tsing Hua University, Hsinchu City 300, Taiwan, ROC.

E-mail address: cychang@math.cts.nthu.edu.tw.

1 The author was supported by NCTS postdoctoral fellowship.

0022-314X/$ – see front matter
doi:10.1016/j.jnt.2008.10.012

lie in a finite extension of k_∞. Finally for a Laurent series $f = \sum_i a_i t^i \in \mathbb{C}_\infty((t))$ and an integer $n \in \mathbb{Z}$, we set $f^{(n)} := \sum_i a_i^{q^n} t^i$ and extend the operation $f \mapsto f^{(n)}$ entrywise to matrices whose entries are in $\mathbb{C}_\infty((t))$.

In 2004, Anderson, Brownawell and Papanikolas [2] developed a criterion for linear independence over function fields, the so-called ABP criterion, to deal with the special values of the geometric Γ-function over A. As a break through, they proved that all the algebraic relations among those special Γ-values are explained by the standard functional equations. Now, we state the ABP criterion as the following:

Theorem 1.1 *(Anderson–Brownawell–Papanikolas). Fix a matrix $\Phi = \Phi(t) \in \mathrm{Mat}_\ell(\bar{k}[t])$ such that $\det \Phi$ is a polynomial in t vanishing (if at all) only at $t = \theta$. Fix a (column) vector $\psi = \psi(t) \in \mathrm{Mat}_{\ell \times 1}(\mathbb{E})$ satisfying the functional equation $\psi^{(-1)} = \Phi\psi$. Evaluating ψ at $t = \theta$, thus obtaining a column vector $\psi(\theta) \in \mathrm{Mat}_{\ell \times 1}(\overline{k_\infty})$. For every (row) vector $\rho \in \mathrm{Mat}_{1 \times \ell}(\bar{k})$ such that $\rho\psi(\theta) = 0$ there exists a (row) vector $P = P(t) \in \mathrm{Mat}_{1 \times \ell}(\bar{k}[t])$ such that $P(\theta) = \rho$, $P\psi = 0$.*

In other words, in the situation of Theorem 1.1, every $\bar{k}$-linear relation among entries of the specialization $\psi(\theta)$ is explained by a $\bar{k}[t]$-linear relation among entries of ψ itself. The main theorem of this paper is the following:

Theorem 1.2. *Fix a matrix $\Phi = \Phi(t) \in \mathrm{Mat}_\ell(\bar{k}[t])$ such that $\det \Phi$ is a polynomial in t satisfying $\det \Phi(0) \neq 0$. Fix a vector $\psi = [\psi_1(t), \ldots, \psi_\ell(t)]^{\mathrm{tr}} \in \mathrm{Mat}_{\ell \times 1}(\mathbb{E})$ satisfying the functional equation $\psi^{(-1)} = \Phi\psi$. Let $\xi \in \bar{k}^\times \backslash \overline{\mathbb{F}_q}^\times$ satisfy*

$$\det \Phi\left(\xi^{(-i)}\right) \neq 0 \quad \textit{for all } i = 1, 2, 3, \ldots.$$

Then we have:

(1) *For every vector $\rho \in \mathrm{Mat}_{1 \times \ell}(\bar{k})$ such that $\rho\psi(\xi) = 0$ there exists a vector $P = P(t) \in \mathrm{Mat}_{1 \times \ell}(\bar{k}[t])$ such that $P(\xi) = \rho$, $P\psi = 0$.*
(2) $\mathrm{tr.deg}_{\bar{k}(t)}\, \bar{k}(t)(\psi_1(t), \ldots, \psi_\ell(t)) = \mathrm{tr.deg}_{\bar{k}}\, \bar{k}(\psi_1(\xi), \ldots, \psi_\ell(\xi))$.

In the situation of above theorem, we note that for any ψ in $\mathrm{Mat}_{\ell \times 1}(\mathbb{T})$ satisfying $\psi^{(-1)} = \Phi\psi$, by Proposition 3.1.3 of [2] the condition $\det \Phi(0) \neq 0$ implies $\psi \in \mathrm{Mat}_{\ell \times 1}(\mathbb{E})$.

Theorem 1.2(1) is an extension of ABP criterion. Theorem 1.2(2) is a consequence of Theorem 1.2(1). It can be thought of as a function field analogue of the Siegel–Shidlovskii theorem concerning E-functions satisfying linear differential equations:

Theorem 1.3 *(Siegel–Shidlovskii, 1956). Let $f_1, \ldots, f_n$ be a set of E-functions which satisfy the system of first-order equations*

$$\frac{d}{dz}\begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} = B \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix},$$

where B is an $n \times n$ matrix with entries in $\overline{\mathbb{Q}}(z)$. Denote the common denominator of the entries of B by $T(z)$. Then, for any $\xi \in \overline{\mathbb{Q}}$ such that $\xi T(\xi) \neq 0$,

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}(z)}\, \overline{\mathbb{Q}}\big(z, f_1(z), \ldots, f_n(z)\big) = \mathrm{tr.deg}_{\overline{\mathbb{Q}}}\, \overline{\mathbb{Q}}\big(f_1(\xi), \ldots, f_n(\xi)\big).$$