Journal of Pure and Applied Algebra 221 (2017) 499–564

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

# Fibrations and Yoneda's lemma in an $\infty$-cosmos

Emily Riehl [a,*], Dominic Verity [b]

[a] *Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218, USA*
[b] *Centre of Australian Category Theory, Macquarie University, NSW 2109, Australia*

ARTICLE INFO

*Article history:*
Received 14 October 2015
Received in revised form 13 June 2016
Available online 29 July 2016
Communicated by J. Adámek

ABSTRACT

We use the terms $\infty$-*categories* and $\infty$-*functors* to mean the objects and morphisms in an $\infty$-*cosmos*: a simplicially enriched category satisfying a few axioms, reminiscent of an enriched category of fibrant objects. Quasi-categories, Segal categories, complete Segal spaces, marked simplicial sets, iterated complete Segal spaces, $\theta_n$-spaces, and fibered versions of each of these are all $\infty$-categories in this sense. Previous work in this series shows that the basic category theory of $\infty$-categories and $\infty$-functors can be developed only in reference to the axioms of an $\infty$-cosmos; indeed, most of the work is internal to the *homotopy 2-category*, a strict 2-category of $\infty$-categories, $\infty$-functors, and natural transformations. In the $\infty$-cosmos of quasi-categories, we recapture precisely the same category theory developed by Joyal and Lurie, although our definitions are 2-categorical in natural, making no use of the combinatorial details that differentiate each model.

In this paper, we introduce cartesian fibrations, a certain class of $\infty$-functors, and their groupoidal variants. Cartesian fibrations form a cornerstone in the abstract treatment of "category-like" structures a la Street and play an important role in Lurie's work on quasi-categories. After setting up their basic theory, we state and prove the Yoneda lemma, which has the form of an equivalence between the quasi-category of maps out of a representable fibration and the quasi-category underlying the fiber over its representing element. A companion paper will apply these results to establish a calculus of *modules* between $\infty$-categories, which will be used to define and study pointwise Kan extensions along $\infty$-functors.



## 1. Introduction

$(\infty, 1)$-*categories* are infinite-dimensional categories with non-invertible morphisms only in dimension one. Equivalently, $(\infty, 1)$-categories are categories weakly enriched over $\infty$-groupoids, i.e., topological spaces. These schematic definitions are realized by a number of concrete *models* of $(\infty, 1)$-categories. Important independent work of Töen and of Barwick and Schommer-Pries proves that all models of $(\infty, 1)$-categories "have the same homotopy theory," in the sense of being connected by a zig-zag of Quillen equivalences of

* Corresponding author.
*E-mail addresses:* eriehl@math.jhu.edu (E. Riehl), dominic.verity@mq.edu.au (D. Verity).

http://dx.doi.org/10.1016/j.jpaa.2016.07.003
0022-4049/

model categories [1] or having equivalent quasi-categories [2]. Inspired by this result, the dream is to be able to work with $(\infty, 1)$-categories “model independently,” which begs the question: can the category theory, and not just the homotopy theory, of $(\infty, 1)$-categories be developed model independently?

This paper describes one possible direction to take in pursuit of that goal. We introduce the notion of an $\infty$-*cosmos*, a simplicially enriched category whose objects we call $\infty$-*categories* and whose morphisms we call $\infty$-*functors* or simply *functors*. A quotient defines a strict 2-category which we call the *homotopy 2-category* of the $\infty$-cosmos, whose objects are again $\infty$-categories, whose morphisms are functors between them, and whose 2-cells are natural transformations of a suitable variety. The homotopy 2-category should be thought of as a categorification of the usual notion of homotopy category spanned by the fibrant-cofibrant objects in a model category that is analogous to the 2-category of ordinary categories, functors, and natural transformations — which indeed is the homotopy 2-category of a suitable $\infty$-cosmos.

Previous work [3–5] shows that a large portion of the category theory of quasi-categories—one model of $(\infty, 1)$-categories that has been studied extensively by Joyal, Lurie, and others—can be developed in the homotopy 2-category of the $\infty$-cosmos of quasi-categories. Indeed, nearly all of the results in these papers, which develop the basic theory of adjunctions, limits and colimits, and monadicity, apply in the homotopy 2-category of any $\infty$-cosmos. In particular, complete Segal spaces, Segal categories, and marked simplicial sets all have their own $\infty$-cosmoi; not coincidentally, these are the models of $(\infty, 1)$-categories whose model categories are the best behaved. Thus each of these varieties of $(\infty, 1)$-categories are examples of $\infty$-categories, in our sense. The axioms imply that the 2-categorical notion of equivalence, interpreted in the homotopy 2-category, precisely coincides with the model categorical notion of weak equivalence. Thus the category theory developed here is appropriately “homotopical,” i.e., weak equivalence invariant.

Unlike the work of Töen and Barwick–Schommer-Pries, an $\infty$-cosmos is not meant to axiomatize a “simplicially enriched category of $(\infty, 1)$-categories.” For instance, slices of an $\infty$-cosmos again define an $\infty$-cosmos. Indeed, $\theta_n$-spaces and iterated complete Segal spaces, two of the most prominent models of $(\infty, n)$-categories, also define $\infty$-cosmoi. Thus, our work begins to develop the basic category theory of $(\infty, n)$-categories as well.

There is a good notion of functor between $\infty$-cosmoi that preserves all of the structure specified by the axiomatization. Examples include “underlying $(\infty, 1)$-category” functors from the cosmoi for $\theta_n$-spaces or iterated complete Segal spaces to the $\infty$-cosmos for quasi-categories. There is also a functor from the $\infty$-cosmos for strict 1-categories (whose homotopy 2-category is the usual 2-category of categories) to the $\infty$-cosmos of quasi-categories or of complete Segal spaces, and also a functor from the $\infty$-cosmos for Kan complexes to the $\infty$-cosmos for quasi-categories. A certain special class of functors of $\infty$-cosmoi, coming from enriched right Quillen equivalences of model categories, both preserve the structures in the $\infty$-cosmoi and reflect equivalences. These functors give a strong meaning to the sense in which the basic category theory of $(\infty, 1)$-categories developed in this framework is “model independent”: basic categorical notions are both preserved and reflected by the functors between the $\infty$-cosmoi of quasi-categories, complete Segal spaces, Segal categories, and marked simplicial sets. Furthermore, a theorem of Low implies that the induced 2-functors between their homotopy 2-categories define bicategorical equivalences [6].

In §2 we define $\infty$-cosmoi and functors between them and demonstrate that all of the examples listed above can be realized as the underlying 1-category of a suitable $\infty$-cosmos. In §3, we define the homotopy 2-category of an $\infty$-cosmos and explore its relevant 2-categorical structure. In fact, the majority of the results in this paper can be stated and proven in an *abstract homotopy 2-category*, which we define to be a (strict) 2-category equipped with *comma objects* and *iso-comma objects* of a suitably weak variety.

The second half of this paper continues the project of developing the basic category theory of $\infty$-categories — including, for the reasons just explained, models of $(\infty, n)$-categories and their sliced variants — by introducing a suitably model independent notion of cartesian fibration, an important cornerstone in an abstract theory of “category-like” structures [7]. Functors valued in $\infty$-categories are most efficiently encoded as *cartesian* or *cocartesian fibrations*, the “co” signaling that the functor so-encoded is covariant. We also study