Journal of Pure and Applied Algebra 221 (2017) 588–610

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Étoiles and valuations

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ARTICLE INFO

Article history:
Received 29 September 2015
Received in revised form 15 June 2016
Available online 27 July 2016
Communicated by A.V. Geramita

MSC:
Primary: 14B25; 14E15; secondary: 13B10; 32B99

ABSTRACT

We establish some properties of étoiles and associated valuations over complex analytic spaces, establishing that Abhyankar's inequality holds. We give some examples of pathological behavior of these valuations. We prove a regularization theorem for complex analytic morphisms. The property of a morphism being regular and the regularization of a morphism play a major role in this theory.

1. Introduction

A local blow up of an analytic space X is a blow up $\pi : X' \to U$ where U is an open subset of X (in the Euclidean topology) and π is the blow up of a closed analytic subspace of U. (An inclusion of an open subset U of X into X is a special case.)

Hironaka defined in [25] and [24] an étoile e over an analytic space X as a subcategory of the category of finite sequences of local blowups over X which satisfies certain good properties. In particular, to each $\pi : X' \to X \in e$ there is an associated point $e_{X'} \in X'$, and given a factorization

$$X' = X_n \xrightarrow{\pi_n} X_{n-1} \to \cdots \to X_1 \xrightarrow{\pi_1} X$$

by local blow ups, we have $\pi_i(e_{X_i}) = e_{X_{i-1}}$ for all i.

In the situation of algebraic geometry (the category of algebraic blowups of an algebraic variety X over a field k) an étoile e can be represented by a valuation of the function field $k(X)$ of X which dominates the local ring $\mathcal{O}_{X,e_X}$ (whose quotient field is $k(X)$). This is the original approach of Zariski [33].

The notion of an étoile e on a complex analytic space X cannot be immediately modeled in valuation theory, even when X is irreducible and nonsingular, as there exist $\pi : X' \to X \in e$ such that X' is not

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[1] Partially supported by NSF grant DMS-1360564.

http://dx.doi.org/10.1016/j.jpaa.2016.07.007
0022-4049/

locally irreducible, and even when X' is locally irreducible, $\mathcal{O}_{X',e_{X'}}$ is generally a very big extension field of $\mathcal{O}_{X,e_X}$.

Valuation theory is an important tool in the birational geometry of algebraic varieties, and it is useful to know which parts of the classical theory for algebraic function fields extend to étoiles on an irreducible nonsingular complex analytic space.

In Section 6 we associate to an étoile over a reduced complex analytic space X a valuation $\nu = \nu_e$ on a giant field which depends on the étoile e. The valuation ring V_e is constructed by taking the union of $\mathcal{O}_{X',e_{X'}}$ where $X' \to X \in e$ is a sequence of local blow ups from nonsingular varieties. We establish in Section 6 that we have, as in the classical case of valuations of algebraic function fields, that

$$\text{rank } \nu \le \text{ratrank } \nu \le \dim X,$$

and if the rational rank ratrank $\nu = \dim X$, then the value group Γ_ν of ν is isomorphic (as an unordered group) to $\mathbb{Z}^{\dim X}$.

The residue field of the valuation ring V_e associated to an étoile e is always isomorphic to $\mathbb{C}$. Thus ν is always a zero dimensional valuation and we see that Abhyankar's inequality [1] for a valuation of a field K which dominates a Noetherian local ring whose quotient field is K, holds for the valuation ν_e associated to an étoile e.

Unlike in the case of algebraic function fields, a composite valuation which arises from an étoile can be very badly behaved, as is shown in the following example. The existence of examples of this type was a major obstruction to a proof of local monomialization of analytic morphisms.

Example 1.1. (Example 7.1) There exists an étoile e on $Y_0 = \mathbb{C}^4$ such that the valuation ring V_e has a proper prime ideal Q such that there exists an infinite chain of local blow ups (of a point in Y_m if m is even and of a nonsingular surface if m is odd)

$$\cdots \to Y_m \to \cdots \to Y_1 \to Y_0$$

with $Y_m \to Y_0 \in e$ for all m, such that the center of Q on Y_m has dimension 3 if m is even and the center of Q on Y_m has dimension 2 if m is odd.

The construction begins with an example by Hironaka, Lejeune and Teissier [26] of a germ of an analytic map $\varphi : (S,a) \to (V,b)$ from a surface to a 3-fold such that no functions in $\mathcal{O}_{V,b}$ vanish on the image of φ but the image becomes a two dimensional analytic subvariety (a surface) after blowing up b.

Hironaka ([25] and [24]) defines La Voûte Étoilée as

$$\mathcal{E}_X = \text{ set of all étoiles over } X \text{ with a topology making } P_X : \mathcal{E}_X \to X,\, e \mapsto e_X \text{ continuous.}$$

Hironaka proves that p_X is proper. This theorem is a generalization of Zariski's theorem [31] on the quasi-compactness of the Zariski Riemann manifold of an algebraic function field.

If $\varphi : Y \to X$ is a dominant morphism of algebraic varieties over a field k (the Zariski closure of $\varphi(Y)$ in X is equal to X) then we have a natural inclusion of algebraic function fields $k(X) \to k(Y)$. Thus a valuation of $k(Y)$ restricts to a valuation of $k(X)$ and a valuation of $k(X)$ can be extended to a valuation of $k(Y)$ (Chapter VI [32]).

However, the situation is much more subtle in the case of complex analytic morphisms of complex analytic spaces (as is exploited in the construction of the above example).

The most useful generalization of the notion of a dominant morphism of algebraic varieties to analytic morphisms of complex analytic spaces is a regular morphism.