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Simple transitive 2-representations of Soergel bimodules in type B_2

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ABSTRACT

We prove that every simple transitive 2-representation of the fiat 2-category of Soergel bimodules (over the coinvariant algebra) in type B_2 is equivalent to a cell 2-representation. We also describe some general properties of the 2-category of Soergel bimodules for arbitrary finite dihedral groups.

1. Introduction

Understanding the approach of categorification via 2-representation theory of 2-categories has its roots in the papers [1,2]. In 2010, Mazorchuk and Miemietz started the series [15–20] of papers in which they systematically study the 2-representation theory of finitary and fiat 2-categories. The latter can be thought of as analogues of finite dimensional algebras (for finitary 2-categories) or finite dimensional algebras with involution (for fiat 2-categories). The first paper [15] in the series introduces the notion of *cell* 2*-representations* as a possible candidate for the notion of "simple" 2-representations in this setting.

Cell 2-representations are inspired by Kazhdan–Lusztig cell modules for Hecke algebras, as defined in [11]. In the case of 2-categories, a left (right, two-sided) cell is a subset of indecomposable 1-morphisms which generate the same left (resp. right, two-sided) ideal. Given a left cell, the corresponding cell 2-representation is a suitable subquotient of the principal 2-representation associated with this left cell. In [15] it was shown that cell 2-representations have a lot of properties similar to properties of usual simple representations.

The paper [19] introduces the notion of *simple transitive* 2*-representations*. By a *transitive* 2-representation one means an additive 2-representation for which the action of 1-morphisms is transitive in the sense that, starting from any indecomposable object and applying all 1-morphisms, one obtains the whole underlying category of the 2-representation by taking the additive closure. A transitive 2-representation is called *simple transitive* if, in addition to the above defined notion of transitivity, we have that the maximal ideal of the 2-representation which is invariant under the 2-action is zero. In other words, the notion of simple

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transitivity has two layers, were the first layer addresses the level of 1-morphisms and the second layer addresses the level of 2-morphisms.

By construction, all cell 2-representations are simple transitive but it is not obvious whether each simple transitive 2-representation is equivalent to a cell 2-representation. As it turns out, in general, these two classes of 2-representations are different. Indeed, in [19] one finds an example of a 2-category for which there exist simple transitive 2-representations which are not cell 2-representations. One of the main results of [19] is that, if the cell structure of the underlying 2-category is "nice enough", then these two classes of 2-representations coincide.

One of the main examples for such a "nice" 2-category is the 2-category of Soergel bimodules over the coinvariant algebra of type A. Hence, for this 2-category, each simple transitive 2-representation is equivalent to a cell 2-representation. Moreover, from [15] it is also known that, for Soergel bimodules in type A, two cell 2-representations corresponding to left cells inside the same two-sided cell are equivalent. For all other types, the cell structure is more complicated and no classification of simple transitive 2-representations is known.

In the present paper we study simple transitive 2-representations of the 2-category of Soergel bimodules over the coinvariant algebra of type B_2. This is the smallest case for which the cell structure does not satisfy the requirements of [19, Theorem 18]. Our main result is the following (see Theorem 6.1):

Theorem A. *Each simple transitive* 2*-representation of the* 2*-category of Soergel bimodules over the coinvariant algebra of type* B_2 *is equivalent to a cell* 2*-representation.*

However, in contrast to type A, we will show that, in type B_2, the cell 2-representations corresponding to the two different left cells inside the unique non-singleton two-sided cell are not equivalent. Behind this phenomenon is the fact that the Kazhdan–Lusztig cell representations to which these different cell 2-representations decategorify are not isomorphic. The latter is due to the fact that these Kazhdan–Lusztig cell representations contain non-isomorphic one-dimensional subquotients. Additionally to the main result formulated above, we study the 2-category $\mathscr{S}_n$ of Soergel bimodules for an arbitrary finite dihedral group D_n and classify for this 2-category all simple transitive 2-representations of small ranks (namely, ranks one and two), generalizing [19, Proposition 21]. Here, the *rank* of an additive 2-representation is the number of isomorphism classes of indecomposable additive generators of the underlying category of the 2-representation.

The proof of the main theorem, which will be given in Section 6, can be divided in two parts. In the first part we study the decategorification of a given simple transitive 2-representation of $\mathscr{S}_4$ and show that all simple transitive 2-representations have either rank one or rank three. In this part we benefit from the fact that the representation theory of the dihedral group is well-known and quite easily described. Another crucial result that we use is the classical Perron–Frobenius Theorem describing the structure of real matrices with positive or non-negative entries.

What is left to show then is that every simple transitive 2-representation of $\mathscr{S}_4$ of rank one or three is equivalent to a cell 2-representation. The rank one case is a bit easier and is treated in Section 4.6. The rank three case, on the other hand, is more involved and constitutes the second part of the proof of the main theorem. It is proved by giving an explicit construction of an equivalence.

The article is organized as follows. In the next section we collect all preliminaries about 2-categories, define fiat and finitary 2-categories and their decategorifications. Section 3 describes the combinatorics of 2-categories; more precisely, the notion of cells of 2-categories is defined and cell 2-representations and simple transitive 2-representations are introduced. In Section 4, we define Soergel bimodules and describe the cell structure of the 2-category of Soergel bimodules for the dihedral group D_n. In particular, these introductory sections summarize definitions and necessary results from [15–20]. Section 5 collects some preliminary results on simple transitive 2-representations of $\mathscr{S}_n$. In Section 6 we classify all simple transitive 2-representations of $\mathscr{S}_4$. Finally, in Section 7 we examine the situation for $n \geq 5$ and prove that every simple transitive