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Residue fields for a class of rational $\mathbf{E}_\infty$-rings and applications

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A R T I C L E I N F O

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A B S T R A C T

Let A be an $\mathbf{E}_\infty$-ring over the rational numbers. If A satisfies a noetherian condition on its homotopy groups $\pi_*(A)$, we construct a collection of $\mathbf{E}_\infty$-A-algebras that realize on homotopy the residue fields of $\pi_*(A)$. We prove an analog of the nilpotence theorem for these residue fields. As a result, we are able to give a complete algebraic description of the Galois theory of A and of the thick subcategories of perfect A-modules. We also obtain partial information on the Picard group of A.

1. Introduction

1.1. Motivation

The goal of this paper is to describe certain invariants of structured ring spectra in characteristic zero. We start by first reviewing the motivation from stable homotopy theory.

The chromatic picture of stable homotopy theory identifies a class of "residue fields" which play an important role in global phenomena. Consider the following ring spectra:

(1) $H\mathbb{Q}$: rational homology.
(2) For each prime p, mod p homology $H\mathbb{F}_p$.
(3) For each prime p and height n, the nth Morava K-theory $K(n)$.

These all define multiplicative homology theories on the category of spectra satisfying *perfect* Künneth isomorphisms: they behave like fields. Moreover, as a consequence of the deep nilpotence technology of [10,17], they are powerful enough to describe much of the structure of the stable homotopy category. For example, one has the following result:

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Theorem 1.1 *(Hopkins–Smith [17]). Let R be a ring spectrum and let $\alpha \in \pi_*(R)$. Then α is nilpotent if and only if the Hurewicz image of α in $\pi_*(F \otimes R)$ is nilpotent, as F ranges over all the ring spectra above.*

This fundamental result was used in [17] to classify the *thick subcategories* of the category of finite p-local spectra for a fixed prime p: all thick subcategories are defined by vanishing conditions for the various residue fields. One can attempt to ask such questions not only for spectra but for general symmetric monoidal, stable ∞-categories, as Hovey, Palmieri, and Strickland have considered in [16]; whenever one has an analog of Theorem 1.1, it is usually possible to prove results along these lines.

For instance, let A be an $\mathbf{E}_\infty$-ring. Then one can try to study such questions in the ∞-category $\mathrm{Mod}(A)$ of A-modules. If $\pi_*(A)$ is concentrated in even degrees and is *regular noetherian,* then it is possible to construct residue fields, prove an analog of Theorem 1.1, and obtain a purely algebraic description of the thick subcategories of perfect A-modules. This has been observed independently by a number of authors. For $\mathbf{E}_\infty$-rings (such as the $\mathbf{E}_\infty$-ring TMF of periodic topological modular forms) which are "built up" appropriately from such nice $\mathbf{E}_\infty$-rings, it is sometimes possible to construct residue fields as well. We used this to classify thick subcategories for perfect modules over $\mathbf{E}_\infty$-rings such as TMF in [26].

1.2. Statement of results

In this paper, we will study such questions over the *rational numbers.* Let A be a rational $\mathbf{E}_\infty$-ring such that the even homotopy groups $\pi_{\mathrm{even}}(A)$ form a noetherian ring and such that the odd homotopy groups $\pi_{\mathrm{odd}}(A)$ form a finitely generated $\pi_{\mathrm{even}}(A)$-module. We will call such rational $\mathbf{E}_\infty$-rings *noetherian.*

For the statement of our first result, we work with $\mathbf{E}_\infty$-rings containing a unit in degree two. In this case, we will produce, for every prime ideal $\mathfrak{p} \subset \pi_0(A)$, a "residue field" of A, which will be an $\mathbf{E}_\infty$-A-algebra whose homotopy groups form a graded field.

Theorem 1.2 *(Existence of residue fields). Let A be a rational, noetherian $\mathbf{E}_\infty$-ring containing a unit in degree two. Given a prime ideal $\mathfrak{p} \subset \pi_0(A)$, there exists an $\mathbf{E}_\infty$-A-algebra $\kappa(\mathfrak{p})$ such that $\kappa(\mathfrak{p})$ is even periodic and the map $\pi_0(A) \to \pi_0(\kappa(\mathfrak{p}))$ induces the reduction $\pi_0 A \to \pi_0(A)_{\mathfrak{p}}/\mathfrak{p}\pi_0(A)_{\mathfrak{p}}$. $\kappa(\mathfrak{p})$ is unique up to homotopy as an object of the ∞-category $\mathrm{CAlg}_{A/}$ of $\mathbf{E}_\infty$-rings under A.*

We will prove an analog of Theorem 1.1 in $\mathrm{Mod}(A)$ for these residue fields.

Theorem 1.3 *(Nilpotence). Suppose A is as above, and let B be an A-ring spectrum; that is, an algebra object in the homotopy category $\mathrm{Ho}(\mathrm{Mod}(A))$. Let $x \in \pi_*(B)$. Then x is nilpotent if and only if for every prime ideal $\mathfrak{p} \subset \pi_0(A)$, the image of x in $\pi_*(B \otimes_A \kappa(\mathfrak{p}))$ is nilpotent.*

The proof of Theorem 1.3 uses entirely different (and much simpler) techniques than Theorem 1.1. However, the conclusion is similar, and we thus find $\mathrm{Mod}(A)$ as an interesting new example of an "axiomatic stable homotopy theory" [16] where many familiar techniques can be applied.

In particular, from Theorem 1.3, we will deduce a classification of thick subcategories of the ∞-category $\mathrm{Mod}^\omega(A)$ of perfect A-modules, for A rational noetherian (not necessarily containing a unit in degree two). Let $\pi_{\mathrm{even}}(A) = \bigoplus_{i \in 2\mathbb{Z}} \pi_i(A)$; this is a graded ring, so $\mathrm{Spec}\pi_{\mathrm{even}}(A)$ inherits a $\mathbb{G}_m$-action.

Theorem 1.4 *(Thick subcategory theorem). Let A be a rational, noetherian $\mathbf{E}_\infty$-ring. The thick subcategories of $\mathrm{Mod}^\omega(A)$ are in natural correspondence with the subsets of the collection of homogeneous prime ideals of $\pi_{\mathrm{even}}(A)$ which are closed under specialization or, equivalently, specialization-closed subsets of the topological space associated to the stack $(\mathrm{Spec}\pi_{\mathrm{even}}(A))/\mathbb{G}_m$.*