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Conditions for the Yoneda algebra of a local ring to be generated in low degrees ☆

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ABSTRACT

The powers $\mathfrak{m}^n$ of the maximal ideal $\mathfrak{m}$ of a local Noetherian ring R are known to satisfy certain homological properties for *large* values of n. For example, the homomorphism $R \to R/\mathfrak{m}^n$ is Golod for $n \gg 0$. We study when such properties hold for *small* values of n, and we make connections with the structure of the Yoneda Ext algebra, and more precisely with the property that the Yoneda algebra of R is generated in degrees 1 and 2. A complete treatment of these properties is pursued in the case of compressed Gorenstein local rings.

0. Introduction

Let $(R, \mathfrak{m}, k)$ be a *local ring*, that is, a commutative noetherian ring R with unique maximal ideal $\mathfrak{m}$ and $k = R/\mathfrak{m}$. For $n \geq 1$ we let $\nu_n \colon \mathfrak{m}^n \to \mathfrak{m}^{n-1}$ denote the canonical inclusion and for each $i \geq 0$ we consider the induced maps

$$\operatorname{Tor}_i^R(\nu_n, k)\colon \operatorname{Tor}_i^R(\mathfrak{m}^n, k) \to \operatorname{Tor}_i^R(\mathfrak{m}^{n-1}, k)\,.$$

Using the terminology of [2], we say that $\mathfrak{m}^n$ is a *small* submodule of $\mathfrak{m}^{n-1}$ if $\operatorname{Tor}_i^R(\nu_n, k) = 0$ for all $i \geq 0$. This condition implies that the canonical projection $\rho_n \colon R \to R/\mathfrak{m}^n$ is a Golod homomorphism, but the converse may not hold.

Levin [9] showed that $\mathfrak{m}^n$ is a small submodule of $\mathfrak{m}^{n-1}$ for all sufficiently large values of n. On the other hand, the fact that $\mathfrak{m}^n$ is a small submodule of $\mathfrak{m}^{n-1}$ for small values of n is an indicator of strong homological properties. It is known that $\mathfrak{m}^2$ is a small submodule of $\mathfrak{m}$ if and only if the Yoneda algebra $\operatorname{Ext}_R(k, k)$ is generated in degree 1, cf. [12, Corollary 1]. More generally, we show:

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Theorem 1. *Let $(R, \mathfrak{m}, k)$ be a local ring. Let $\widehat{R} = Q/I$ be a minimal Cohen presentation of R, with $(Q, \mathfrak{n}, k)$ a regular local ring and $I \subseteq \mathfrak{n}^2$. Let t be an integer such that $I \subseteq \mathfrak{n}^t$. The following statements are then equivalent:*

(1) *$\mathfrak{m}^t$ is a small submodule of $\mathfrak{m}^{t-1}$;*
(2) *$\rho_t \colon R \to R/\mathfrak{m}^t$ is Golod;*
(3) *$\rho_n \colon R \to R/\mathfrak{m}^n$ is Golod for all n such that $t \le n \le 2t - 2$;*
(4) *$I \cap \mathfrak{n}^{t+1} \subseteq \mathfrak{n}I$ and the algebra $\mathrm{Ext}^*_R(k, k)$ is generated by $\mathrm{Ext}^1_R(k, k)$ and $\mathrm{Ext}^2_R(k, k)$.*

If R is artinian, its *socle degree* is the largest integer s with $\mathfrak{m}^s \ne 0$. When R is a compressed Gorenstein local ring (see Section 3 for a definition) of socle degree $s \ne 3$, we determine all values of the integer n for which the homomorphism ρ_n is Golod, respectively for which $\mathfrak{m}^n$ is a small submodule of $\mathfrak{m}^{n-1}$, and we use Theorem 1 to establish part (3) below.

Theorem 2. *Let $(R, \mathfrak{m}, k)$ be a compressed Gorenstein local ring of socle degree s. Assume $2 \le s \ne 3$ and let t denote the smallest integer such that $2t \ge s + 1$. If $n \ge 1$, then the following hold:*

(1) *$\mathfrak{m}^n$ is a small submodule of $\mathfrak{m}^{n-1}$ if and only if $n > s$ or $n = s + 2 - t$.*
(2) *$\rho_n \colon R \to R/\mathfrak{m}^n$ is Golod if and only if $n \ge s + 2 - t$.*
(3) *If s is even, then $\mathrm{Ext}_R(k, k)$ is generated by $\mathrm{Ext}^1_R(k, k)$ and $\mathrm{Ext}^2_R(k, k)$.*

The conclusion of (3) does not hold when s is odd, see Corollary 3.7.

Section 1 provides definitions and properties of the homological notions of interest. Theorem 1 is proved in Section 2 and Theorem 2 is proved in Section 3.

1. Preliminaries

Throughout the paper $(R, \mathfrak{m}, k)$ denotes a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let M be a finitely generated R-module.

We denote by $\widehat{R}$ the completion of R with respect to $\mathfrak{m}$. A *minimal Cohen presentation* of R is a presentation $\widehat{R} = Q/I$, with Q a regular local ring with maximal ideal $\mathfrak{n}$ and I an ideal with $I \subseteq \mathfrak{n}^2$. We know that such a presentation exists, by the Cohen structure theorem.

We denote by R^{g} the associated graded ring with respect to $\mathfrak{m}$, and by M^{g} the associated graded module with respect to $\mathfrak{m}$. We denote by $(R^{\mathrm{g}})_j$ the j-th graded component of R^{g}. For any $x \in R$ we denote by x^* the image of x in $\mathfrak{m}^j/\mathfrak{m}^{j+1} = (R^{\mathrm{g}})_j$, where j is such that $x \in \mathfrak{m}^j \smallsetminus \mathfrak{m}^{j+1}$. For an ideal J of R, we denote by J^* the homogeneous ideal generated by the elements x^* with $x \in J$.

Remark 1.1. With $\widehat{R} = Q/I$ as above, the following then hold:

(1) $I \subseteq \mathfrak{n}^t$ if and only if $\mathrm{rank}_k(\mathfrak{m}^{t-1}/\mathfrak{m}^t) = \binom{e+t-2}{e-1}$, where e denotes the minimal number of generators of $\mathfrak{m}$.
(2) Assume $t \ge 2$ and $I \subseteq \mathfrak{n}^t$. Then $I \cap \mathfrak{n}^{t+1} \subseteq \mathfrak{n}I$ if and only if the map

$$\mathrm{Ext}^2_{\rho_t}(k, k) \colon \mathrm{Ext}^2_{R/\mathfrak{m}^t}(k, k) \to \mathrm{Ext}^2_R(k, k)$$

induced by the canonical projection $\rho_t \colon R \to R/\mathfrak{m}^t$ is surjective.