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Solvability and nilpotency for algebraic supergroups

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ABSTRACT

We study solvability, nilpotency and splitting property for algebraic supergroups over an arbitrary field K of characteristic $\operatorname{char} K \neq 2$. Our first main theorem tells us that an algebraic supergroup $\mathbb{G}$ is solvable if the associated algebraic group $\mathbb{G}_{ev}$ is trigonalizable. To prove it we determine the algebraic supergroups $\mathbb{G}$ such that $\dim \operatorname{Lie}(\mathbb{G})_1 = 1$; their representations are studied when $\mathbb{G}_{ev}$ is diagonalizable. The second main theorem characterizes nilpotent connected algebraic supergroups. A super-analogue of the Chevalley Decomposition Theorem is proved, though it must be in a weak form. An appendix is given to characterize smooth Noetherian superalgebras as well as smooth Hopf superalgebras.

1. Introduction

We work over an arbitrary field K of characteristic $\operatorname{char} K \neq 2$, unless otherwise sated. Our aim is to pursue super-analogues of the following three: (1) solvability of trigonalizable algebraic groups, (2) nilpotency criteria for connected algebraic groups, (3) the Chevalley Decomposition Theorem for affine groups. *Affine groups* (resp., *algebraic groups*) mean what are called affine group schemes (resp., algebraic affine group schemes) in [29].

General references on supersymmetry are [2,4,27].

1.1. Basic definitions

"Super" is a synonym of "graded by the group $\mathbb{Z}_2 = \{0, 1\}$ of order 2". Therefore, *super-vector spaces* are precisely $\mathbb{Z}_2$-graded vector spaces, $V = V_0 \oplus V_1$; the component V_0 (resp., V_1) and its elements are called *even* (resp., *odd*). Those spaces form a symmetric category, SMod_K, with respect to the so-called super-symmetry. The ordinary objects, such as Hopf or Lie algebras, defined in the symmetric category of vector spaces are generalized to the super-objects, such as Hopf or Lie superalgebras, defined in SMod_K.

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Indeed, every ordinary object, say A, is regarded as a special super-object that is *purely even* in the sense $A = A_0$.

An *affine supergroup* is a representable group-valued functor defined on the category SAlg_K of super-commutative superalgebras. Such a functor, say $\mathbb{G}$, is uniquely represented by a super-commutative Hopf superalgebra. We denote this Hopf superalgebra by $K[\mathbb{G}]$. The category Alg_K of commutative algebras, or purely even super-commutative superalgebras, is a full subcategory of SAlg_K. Given an affine supergroup $\mathbb{G}$, the restricted functor

$$\mathbb{G}_{ev} = \mathbb{G}|_{\mathsf{Alg}_K}$$

is an affine group; this is indeed represented by the Hopf algebra $K[\mathbb{G}]/(K[\mathbb{G}]_1)$, where $(K[\mathbb{G}]_1)$ is the Hopf super-ideal of $K[\mathbb{G}]$ generated by the odd component $K[\mathbb{G}]_1$ of $K[\mathbb{G}]$. An affine supergroup $\mathbb{G}$ is called an *algebraic supergroup* if $K[\mathbb{G}]$ is finitely generated as an algebra. The associated $\mathbb{G}_{ev}$ is then an algebraic group.

1.2. Solvability of even-trigonalizable supergroups

An algebraic supergroup $\mathbb{G}$ is said to be *even-trigonalizable* (resp., *even-diagonalizable*) if the algebraic group $\mathbb{G}_{ev}$ is trigonalizable (resp., diagonalizable). Our first main result, Theorem 6.2, states that every even-trigonalizable supergroup $\mathbb{G}$ has a normal chain of closed super-subgroups

$$\mathbb{G}_0 \lhd \mathbb{G}_1 \lhd \ldots \lhd \mathbb{G}_t = \mathbb{G}, \quad t \geq 0$$

such that $\mathbb{G}_0$ is a trigonalizable algebraic group, and each factor $\mathbb{G}_i/\mathbb{G}_{i-1}$ is isomorphic to one of the elementary supergroups G_a^-, G_m and μ_n, $n > 1$; see Example 2.5. As a corollary, every even-trigonalizable supergroup is solvable; this generalizes the classical result for trigonalizable algebraic groups. We have also that a connected smooth algebraic supergroup $\mathbb{G}$ is solvable if and only if $\mathbb{G}_{ev}$ is solvable. These results are proved in Section 6. A key of the proof is to determine those algebraic supergroups $\mathbb{G}$ whose Lie superalgebra $\mathrm{Lie}(\mathbb{G})$ has one-dimensional odd component, or in notation, $\dim \mathrm{Lie}(\mathbb{G})_1 = 1$; this is done in Section 4. Our result presents explicitly such an algebraic supergroup as $G_{g,x}$, parameterizing it by an algebraic group G and elements $g \in K[G]$, $x \in \mathrm{Lie}(G)$ which satisfy some conditions; see Lemma–Definition 4.2. If G is a diagonalizable algebraic group D, then $D_{g,x}$ is even-diagonalizable. In Section 5 we discuss representations of $D_{g,x}$, determining the (injective) indecomposables and the simples. The consequences are used in Section 8 when we discuss (counter-)examples.

1.3. Nilpotency criteria for connected supergroups

These criteria are given by our second main result, Theorem 7.4. In particular, it is proved that a connected algebraic supergroup $\mathbb{G}$ is nilpotent if and only if it fits into a central extension $1 \to F \to \mathbb{G} \to \mathbb{U} \to 1$ of a unipotent supergroup $\mathbb{U}$ by an algebraic group F of multiplicative type. Section 7 is devoted to proving the theorem. An ingredient is Proposition 7.1, which describes the algebraic group $\mathcal{Z}(\mathbb{G})_{ev}$ associated with the center $\mathcal{Z}(\mathbb{G})$ of an algebraic supergroup $\mathbb{G}$.

1.4. Harish–Chandra pairs

To prove our results so far stated, a crucial role will be played by the category equivalence between the algebraic supergroups and the Harish–Chandra pairs. A *Harish–Chandra pair* is a pair (G, V) of an algebraic group G and a finite-dimensional right G-module, given as a structure a G-equivariant bilinear