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Hochschild products and global non-abelian cohomology for algebras. Applications ☆

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ABSTRACT

Let A be a unital associative algebra over a field k, E a vector space and $\pi : E \to A$ a surjective linear map with $V = \mathrm{Ker}(\pi)$. All algebra structures on E such that $\pi : E \to A$ becomes an algebra map are described and classified by an explicitly constructed global cohomological type object $\mathbb{GH}^2(A, V)$. Any such algebra is isomorphic to a Hochschild product $A \star V$, an algebra introduced as a generalization of a classical construction. We prove that $\mathbb{GH}^2(A, V)$ is the coproduct of all non-abelian cohomologies $\mathbb{H}^2(A, (V, \cdot))$. The key object $\mathbb{GH}^2(A, k)$ responsible for the classification of all co-flag algebras is computed. All Hochschild products $A \star k$ are also classified and the automorphism groups $\mathrm{Aut}_{\mathrm{Alg}}(A \star k)$ are fully determined as subgroups of a semidirect product $A^* \ltimes \big(k^* \times \mathrm{Aut}_{\mathrm{Alg}}(A)\big)$ of groups. Several examples are given as well as applications to the theory of supersolvable coalgebras or Poisson algebras. In particular, for a given Poisson algebra P, all Poisson algebras having a Poisson algebra surjection on P with a 1-dimensional kernel are described and classified.

0. Introduction

Introduced at the level of groups by Hölder [28], the extension problem is a famous and still open problem to which a vast literature was devoted (see [1] and the references therein). Fundamental results obtained for groups [1,15,37] served as a model for studying the extension problem for several other fields such as Lie/Leibniz algebras [14,31], super Lie algebras [6], associative algebras [17,26], Hopf algebras [8], Poisson algebras [24], Lie–Rinehart algebras [12,25], etc. The extension problem is one of the main tools

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for classifying 'finite objects' and has been a source of inspiration for developing cohomology theories in all fields mentioned above. We recall the extension problem using the language of category theory. Let $\mathcal{C}$ be a category having a zero (i.e. an initial and final) object 0 and for which it is possible to define an exact sequence. Given A, B two *fixed* objects of $\mathcal{C}$, the extension problem consists of the following question:

Describe and classify all *extensions of A by B*, i.e. all triples (E, i, π) consisting of an object E of $\mathcal{C}$ and two morphisms in $\mathcal{C}$ that fit into an exact sequence of the form:

$$0 \longrightarrow B \xrightarrow{\ i\ } E \xrightarrow{\ \pi\ } A \longrightarrow 0$$

Two extensions (E, i, π) and (E', i', π') of A by B are called *equivalent* if there exists an isomorphism $\varphi : E \to E'$ in $\mathcal{C}$ that stabilizes B and co-stabilizes A, i.e. $\varphi \circ i = i'$ and $\pi' \circ \varphi = \pi$. The answer to the extension problem is given by explicitly computing the set $\mathrm{Ext}(A, B)$ of all equivalence classes of extensions of A by B via this equivalence relation. The simplest case is that of extensions with an 'abelian' kernel B for which a Schreier type theorem proves that all extensions of A by an abelian object B are classified by the second cohomology group $\mathrm{H}^2(A, B)$ – the result is valid for groups, Lie/Leibniz/associative/Poisson/Hopf algebras, but the construction of the second cohomology group is different for each of the above categories [8,14,15,26,37]. The difficult part of the extension problem is the case when B is not abelian: as a general principle, the Schreier type theorems remain valid, but this time the classifying object of all extensions of A by B is not the cohomology group of a given complex anymore, but only a pointed set called the *non-abelian cohomology* $\mathrm{H}^2_{\mathrm{nab}}(A, B)$. For its construction in the case of groups we refer to [7], while for Lie algebras to [19] where it was proved that the non-abelian cohomology $\mathrm{H}^2_{\mathrm{nab}}(A, B)$ is the Deligne groupoid of a suitable differential graded Lie algebra. The difficulty of the problem consists in explicitly computing $\mathrm{H}^2_{\mathrm{nab}}(A, B)$: lacking an efficient cohomology tool as in the abelian case [35,38], it needs to be computed 'case by case' using different computational and combinatorial approaches.

This paper deals, at the level of associative algebras, with a generalization of the extension problem, called the *global extension (GE) problem*, introduced recently in [3,33] for Poisson/Leibniz algebras as a categorical dual of the extending structures problem [2,4,5]. The GE-problem can be formulated for any category $\mathcal{C}$ using a simple idea: in the classical extension problem we drop the hypothesis 'B is a fixed object in $\mathcal{C}$' and replace it by a weaker one, namely 'B has a fixed dimension'. For example, if $\mathcal{C}$ is the category of unital associative algebras over a field k, the GE-problem can be formulated as follows: for a given algebra A, classify all associative algebras E for which there exists a surjective algebra map $E \to A \to 0$ whose kernel has a given dimension $\mathfrak{c}$ as a vector space. Of course, any such algebra has $A \times V$ as the underlying vector space, where V is a vector space such that $\dim(V) = \mathfrak{c}$. Among several equivalent possibilities for writing down the GE-problem for algebras, we prefer the following:

Let A be a unital associative algebra, E a vector space and $\pi : E \to A$ a linear epimorphism of vector spaces. Describe and classify the set of all unital associative algebra structures that can be defined on E such that $\pi : E \to A$ becomes a morphism of algebras.

By classification of two algebra structures $\cdot_E$ and $\cdot'_E$ on E we mean the classification up to an isomorphism of algebras $(E, \cdot_E) \cong (E, \cdot'_E)$ that stabilizes $V := \mathrm{Ker}(\pi)$ and co-stabilizes A: we shall denote by $\mathrm{Gext}(A, E)$ the set of equivalence classes of all algebra structures on E such that $\pi : E \to A$ is an algebra map. Let us explain now the significant differences between the GE-problem and the classical extension problem for associative algebras whose study was initiated in [17,26]. Let $(E, \cdot_E)$ be a unital algebra structure on E such that $\pi : (E, \cdot_E) \to A$ is an algebra map. Then $(E, \cdot_E)$ is an extension of the unital algebra A by the associative algebra $V = \mathrm{Ker}(\pi)$, which is a non-unital subalgebra (in fact a two-sided ideal) of $(E, \cdot_E)$. However, the multiplication on V is not fixed from the input data, as in the case of the classical extension problem: it depends essentially on the algebra structures on E which we are looking for. Thus the classical extension problem is in some sense the *'local'* version of the GE-problem. The partial answer and the best result obtained so far for the classical extension problem was given in [26, Theorem 6.2]: all algebra