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Locally finite groups in which every non-cyclic subgroup is self-centralizing

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ABSTRACT

Locally finite groups having the property that every non-cyclic subgroup contains its centralizer are completely classified.

1. Introduction

A subgroup H of a group G is *self-centralizing* if the centralizer $C_G(H)$ is contained in H. In [1] it has been remarked that a locally graded group in which all non-trivial subgroups are self-centralizing has to be finite; therefore it has to be either cyclic of prime order or non-abelian of order being the product of two different primes.

In this article, we consider the more extensive class $\mathfrak{X}$ of all groups in which every non-cyclic subgroup is self-centralizing. In what follows we use the term $\mathfrak{X}$-groups in order to denote groups in the class $\mathfrak{X}$. The study of properties of $\mathfrak{X}$-groups was initiated in [1]. In particular, the first four authors determined the structure of finite $\mathfrak{X}$-groups which are either nilpotent, supersoluble or simple.

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In this paper, Theorem 2.1 gives a complete classification of finite $\mathfrak{X}$-groups. We remark that this result does not depend on classification of the finite simple groups rather only on the classification of groups with dihedral or semidihedral Sylow 2-subgroups. We also determine the infinite soluble $\mathfrak{X}$-groups, and the infinite locally finite $\mathfrak{X}$-groups, the results being presented in Theorems 3.6 and 3.7. It turns out that these latter groups are suitable finite extensions either of the infinite cyclic group or of a Prüfer p-group, $\mathbb{Z}_{p^\infty}$, for some prime p. Theorem 3.7 together with Theorem 2.1 provides a complete classification of locally finite $\mathfrak{X}$-groups.

We follow [2] for basic group theoretical notation. In particular, we note that $F^*(G)$ denotes the generalized Fitting subgroup of G, that is the subgroup of G generated by all subnormal nilpotent or quasisimple subgroups of G. The latter subgroups are the components of G. We see from [2, Section 31] that distinct components commute. The fundamental property of the generalized Fitting subgroup that we shall use is that it contains its centralizer in G [2, (31.13)]. We denote the alternating group and symmetric group of degree n by $\mathrm{Alt}(n)$ and $\mathrm{Sym}(n)$ respectively. We use standard notation for the classical groups. The notation $\mathrm{Dih}(n)$ denotes the dihedral group of order n and Q_8 is the quaternion group of order 8. The term quaternion group will cover groups which are often called generalized quaternion groups. The cyclic group of order n is represented simply by n, so for example $\mathrm{Dih}(12) \cong 2 \times \mathrm{Dih}(6) \cong 2 \times \mathrm{Sym}(3)$. Finally $\mathrm{Mat}(10)$ denotes the Mathieu group of degree 10. The Atlas [3] conventions are used for group extensions. Thus, for example, $p^2{:}\mathrm{SL}_2(p)$ denotes the split extension of an elementary abelian group of order p^2 by $\mathrm{SL}_2(p)$.

2. Finite $\mathfrak{X}$-groups

In this section we determine all the finite groups belonging to the class $\mathfrak{X}$. The main result is the following.

Theorem 2.1. *Let G be a finite $\mathfrak{X}$-group. Then one of the following holds:*

- (1) *If G is nilpotent, then either*
  - (1.1) *G is cyclic;*
  - (1.2) *G is elementary abelian of order p^2 for some prime p;*
  - (1.3) *G is an extraspecial p-group of order p^3 for some odd prime p; or*
  - (1.4) *G is a dihedral, semidihedral or quaternion 2-group.*
- (2) *If G is supersoluble but not nilpotent, then, letting p denote the largest prime divisor of $|G|$ and $P \in \mathrm{Syl}_p(G)$, we have that P is a normal subgroup of G and one of the following holds:*
  - (2.1) *P is cyclic and either*
    - (2.1.1) *$G \cong D \ltimes C$, where C is cyclic, D is cyclic and every non-trivial element of D acts fixed point freely on C (so G is a Frobenius group);*
    - (2.1.2) *$G = D \ltimes C$, where C is a cyclic group of odd order, D is a quaternion group, and $C_G(C) = C \times D_0$ where D_0 is a cyclic subgroup of index 2 in D with G/D_0 a dihedral group; or*
    - (2.1.3) *$G = D \ltimes C$, where D is a cyclic q-group, C is a cyclic q'-group (here q denotes the smallest prime dividing the order of G), $1 < Z(G) < D$ and $G/Z(G)$ is a Frobenius group;*
  - (2.2) *P is extraspecial and G is a Frobenius group with cyclic Frobenius complement of odd order dividing $p - 1$.*
- (3) *If G is not supersoluble and $F^*(G)$ is nilpotent, then either (3.1) or (3.2) below holds.*
  - (3.1) *$F^*(G)$ is elementary abelian of order p^2, $F^*(G)$ is a minimal normal subgroup of G and one of the following holds:*
    - (3.1.1) *$p = 2$ and $G \cong \mathrm{Sym}(4)$ or $G \cong \mathrm{Alt}(4)$; or*