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A calculus of lax fractions ☆

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A R T I C L E I N F O

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A B S T R A C T

We present a notion of category of lax fractions, where lax fraction stands for a formal composition $s_* f$ with $s_* s = \mathrm{id}$ and $ss_* \leq \mathrm{id}$, and a corresponding calculus of lax fractions which generalizes the Gabriel–Zisman calculus of fractions.

1. Introduction

Given a class Σ of morphisms of a category $\mathcal{X}$, we can construct a category of fractions $\mathcal{X}[\Sigma^{-1}]$ where all morphisms of Σ are invertible. More precisely, we can define a functor $P_\Sigma : \mathcal{X} \to \mathcal{X}[\Sigma^{-1}]$ which takes the morphisms of Σ to isomorphisms, and, moreover, P_Σ is universal with respect to this property. As shown in [12], if Σ admits a calculus of fractions, then the morphisms of $\mathcal{X}[\Sigma^{-1}]$ can be expressed by equivalence classes of cospans (f, g) of morphisms of $\mathcal{X}$ with $g \in \Sigma$, which correspond to the formal compositions $g^{-1}f$.

We recall that categories of fractions are closely related to reflective subcategories and orthogonality. In particular, if $\mathcal{A}$ is a full reflective subcategory of $\mathcal{X}$, the class Σ of all morphisms inverted by the corresponding reflector functor – equivalently, the class of all morphisms with respect to which $\mathcal{A}$ is orthogonal – admits a left calculus of fractions; and $\mathcal{A}$ is, up to equivalence of categories, a category of fractions of $\mathcal{X}$ for Σ. In [3] we presented a Finitary Orthogonality Deduction System inspired by the left calculus of fractions, which can be looked as a generalization of the Implicational Logic of [18], see [4].

Assume now that $\mathcal{X}$ is an order-enriched category, that is, its hom-sets $\mathcal{X}(X, Y)$ are endowed with a partial order satisfying the condition $f \leq g \Rightarrow hfj \leq hgj$ for every morphisms $f, g : X \to Y$, $j : Z \to X$ and $h : Y \to W$. We call a morphism $f : X \to Y$ of $\mathcal{X}$ a *left adjoint section* if it is a left adjoint and has a left inverse; equivalently, there is a morphism $f_* : Y \to X$ such that $f_* f = \mathrm{id}_X$ and $ff_* \leq \mathrm{id}_Y$. We are interested in a category of lax fractions in the sense that, given a class Σ of morphisms of $\mathcal{X}$, we want a category $\mathcal{X}[\Sigma_*]$ and an order-enriched functor $P_\Sigma : \mathcal{X} \to \mathcal{X}[\Sigma_*]$ which takes morphisms of Σ to left adjoint

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sections of $\mathcal{X}[\Sigma_*]$ and, moreover, P_Σ is universal with respect to that property. This problem is connected with the study of KZ-monads and Kan-injectivity as explained next.

In recent papers ([1,7]) we have studied a lax version of orthogonality in order-enriched categories: Kan-injectivity. An object A is said to be (left) Kan-injective with respect to a morphism $h : X \to Y$ provided that for every morphism $f : X \to A$ there is a left Kan extension of f along h, denoted f/h, and, moreover, $f = (f/h)h$. And a morphism $k : A \to B$ is said to be Kan-injective with respect to h if A and B are so and k preserves left Kan extensions along h, i.e., $(kf)/h = k(f/h)$. Let $\mathcal{A}$ be a subcategory of an order-enriched category $\mathcal{X}$. We say that $\mathcal{A}$ is KZ-reflective if it is reflective and the monad induced in $\mathcal{X}$ by the reflector functor $F : \mathcal{X} \to \mathcal{A}$ is a KZ-monad, i.e., the unit η satisfies the inequalities $F\eta_X \leq \eta_{FX}$ for all objects X of $\mathcal{X}$ ([16,11]). If, moreover, $\mathcal{A}$ is an Eilenberg–Moore category of a KZ-monad over $\mathcal{X}$, we say that $\mathcal{A}$ is a KZ-monadic subcategory of $\mathcal{X}$. Let $\mathcal{A}^{\mathsf{LInj}}$ denote the class of all morphisms with respect to which all objects and morphisms of $\mathcal{A}$ are left Kan-injective. As shown in [7], if $\mathcal{A}$ is KZ-reflective in $\mathcal{X}$, $\mathcal{A}^{\mathsf{LInj}}$ consists precisely of all morphisms of $\mathcal{X}$ whose images through the reflector functor are left adjoint sections.

In this paper we present the notion of category of lax fractions $P_\Sigma : \mathcal{X} \to \mathcal{X}[\Sigma_*]$ and a calculus of lax fractions which generalize the usual non-lax versions. But now Σ is not just a class of morphisms, as in the ordinary case; instead, it is a subcategory of the arrow category $\mathcal{X}^{\to}$. And the calculus of lax fractions is expressed as a calculus of squares (called Σ-squares) which represent formal equalities of the form $fr_* = s_*g$ (see Section 4). This way, we obtain a description of the category of lax fractions of $\mathcal{X}$, for Σ a subcategory of $\mathcal{X}^{\to}$ admitting a left calculus of lax fractions, in terms of formal fractions s_*f represented by cospans $\bullet \xrightarrow{f} \bullet \xleftarrow{s} \bullet$ with s an object of Σ (Theorem 4.11). The idea of "calculating" with squares of the base category $\mathcal{X}$ instead of just with morphisms of $\mathcal{X}$ is also used in the paper in preparation [2] in order to obtain a Kan-Injectivity Logic generalizing the Orthogonality Logic of [3].

Given a subcategory $\mathcal{A}$ of $\mathcal{X}$, let $\mathcal{A}^{\underline{\mathsf{LInj}}}$ denote the subcategory of $\mathcal{X}^{\to}$ whose objects are the morphisms of $\mathcal{A}^{\mathsf{LInj}}$, and whose morphisms between them are those of the form $(u, v) : (s : X \to Y) \longrightarrow (s' : Z \to W)$ such that $(fu)/s = (f/s')v$ for all f with domain Z and codomain in $\mathcal{A}$. We show that, for $\Sigma = \mathcal{A}^{\underline{\mathsf{LInj}}}$, if $\mathcal{A}$ is a KZ-reflective subcategory of $\mathcal{X}$, the category $\mathcal{X}[\Sigma_*]$ is the Kleisli category for the monad induced by the reflector functor $F : \mathcal{X} \to \mathcal{A}$, and F differs from the functor $P_\Sigma : \mathcal{X} \to \mathcal{X}[\Sigma_*]$ at most by closedness under left adjoint retractions (Theorem 3.7); moreover, Σ admits a left calculus of lax fractions (Proposition 4.5).

We finish up with some properties on cocompleteness. We show that whenever $\mathcal{X}$ has weighted colimits, any subcategory of $\mathcal{X}^{\to}$ of the form $\Sigma = \mathcal{A}^{\underline{\mathsf{LInj}}}$ also has weighted colimits (Theorem 5.1) and admits a left calculus of lax fractions, and the corresponding category of lax fractions $\mathcal{X}[\Sigma_*]$ has (small) conical coproducts. Moreover, we present conditions on any subcategory Σ under which $\mathcal{X}[\Sigma_*]$ has finite conical coproducts, provided $\mathcal{X}$ has them.

Several examples of subcategories Σ of $\mathcal{X}^{\to}$ admitting a left calculus of lax fractions are provided in Examples 4.4 for $\mathcal{X}$ the category Pos of posets and monotone maps, the category Loc of locales and localic maps, and the category Top_0 of T_0 topological spaces and continuous maps.

The study of constructions of categories by freely adding adjoints to the arrows of a category has been addressed before. Although the present approach is completely different, it is worth mentioning here the works [9] and [10] of Dawson, Paré and Pronk.

2. Preliminaries

Along this paper we work in the order-enriched context. More precisely, we consider categories and functors enriched in the category Pos of posets and monotone maps. For a category $\mathcal{X}$ this means that each one of its hom-sets $\mathcal{X}(X, Y)$ is equipped with a partial order $\leq$ which is preserved by composition on the left and on the right. And a functor between order-enriched categories is order-enriched if it preserves the partial order of the morphisms. A subcategory of an order-enriched category $\mathcal{X}$ will be considered order-enriched via the restriction of the order on the morphisms of $\mathcal{X}$ to the morphisms of $\mathcal{A}$.