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Logical systems I: Internal calculi

Michal R. Przybylek

Polish–Japanese Academy of Information Technology, Poland

A R T I C L E   I N F O

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A B S T R A C T

This is the first of a series of papers on a categorical approach to the logical systems. Its aim is to set forth the necessary foundations for more advanced concepts. The paper shows how the internal models of various lambda calculi arise in any 2-category with a notion of discreteness. We generalise to a 2-categorical setting two famous theorems: one saying that under some mild conditions, an object is internally complete iff it is internally cocomplete; and another saying that an object of a sufficiently cocomplete 2-category cannot be internally (co)complete unless it is degenerated. The first of the theorems specialises to a well-known fact from order theory, and also provides non-trivial results about posets and categories in constructive mathematics. Second of the theorems gives a powerful generalisation of Freyd's Theorem and sheds more light on the difficulty of finding fibrational models for higher polymorphism. As a simple corollary of this theorem, we obtain a variant of Freyd's Theorem for the categories internal to any *tensored* category. There is also a hidden objective of the paper — reading it backwards should provide a gentle introduction to the 2-categorical concepts *through* internal categories.

0. Preliminaries

There are mainly two distinct approaches to the logic. One of them is a "model-theoretic" approach. This approach deals with a set of formulae *Sen* and a class of models *Mod* together with a binary relation $\models \subseteq \mathit{Mod} \times \mathit{Sen}$ saying which formulae are true in which models.

Example 0.1 *(Model-theoretic propositional logic).* Let Σ_{Prop} be the propositional signature — that is, the signature consisting of two nullary symbols $\top, \bot$ and three binary symbols $\wedge, \vee, \Rightarrow$. A propositional syntax *Prop* is the free algebra over Σ_{Prop} on a countable set of generators *Var* (the set of variables). Denote by *Bool* the class of pairs $\langle B, \nu \colon \mathit{Var} \to |B| \rangle$, where B is a Boolean algebra over Σ_{Prop}, $|B|$ is its carrier, and $\nu \colon \mathit{Var} \to |B|$ is the valuation of variables *Var* in the carrier of B. By freeness of *Prop*,

E-mail address: mrp@mimuw.edu.pl.

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every valuation $\nu \in |B|^{Var}$ uniquely extends to the homomorphism $\nu^{\#} \in B^{Prop}$. We define model-theoretic classical propositional logic as the relation $\models_B \subseteq Bool \times |Prop|$

$$\langle B, \nu \colon Var \to |B| \rangle \models_B \phi \text{ iff } \nu^{\#}(\phi) = \top_B$$

where $\top_B$ is the interpretation of $\top$ in algebra B.

As usual, we say that a formula ϕ is valid if it is satisfied in every model, that is, for every model B, and every valuation $\nu \in |B|^{Var}$ the following holds:

$$\langle B, \nu \rangle \models_B \phi$$

If we replace Boolean algebras with Heyting algebras in the above definition, we obtain model-theoretic intuitionistic propositional logic:

$$\models_H \subseteq Heyting \times |Prop|$$

Recalling that the 2-valued Boolean algebra $2 = \{0 \leq 1\}$ is complete for classical propositional logic[1] we may reach the following compact characterisation.

Example 0.2 *(Model-theoretic 2-valued logic).* With the notation of Example 0.1 we define model-theoretic 2-valued propositional logic to be the relation $\models_2 \subseteq |2|^{Var} \times |Prop|$

$$\nu \models_2 \phi \text{ iff } \nu^{\#}(\phi) = \top_2$$

In the above example the logical connectives were defined internally to the logic — i.e. were defined inductively over the syntax to imitate the operations from Boolean algebra. Another way is to define the logical connectives externally to the logic. This idea may be found in the theory of specifications, where it is used to give an abstract characterisation of the logical connectives, or to enrich logical systems with some "missing" connectives (Example 4.1.41 of [1]). For example we can extend the set $|Prop|$ by the formulae of the form $\urcorner\phi$ and put:

$$\nu \models \urcorner\phi \text{ iff } \nu \not\models \phi$$

Connectives defined in such a way do not depend on the structure of Boolean algebra 2, but on the connectives from the "external" logic — i.e. the meta-logic that defines the relation $\models$. We shall call these connectives "extensionally defined" as they rely on the logical values of $\models$.

Another approach to the logic is a "proof-theoretic" approach. This approach deals with deductive systems, or categories. Following Lambek and Scott [2] we define a graph G to be a quadruple $\langle Obj; Arr; src, trg \colon Arr \to Obj \rangle$, and shall call the elements of Obj objects (or formulae), the elements of Arr arrows (morphisms, proofs, or deductions) and write $f \colon A \to B \in G$ for $f \in Arr \wedge src(f) = A \wedge trg(f) = B$. Then a deductive system is a graph in which with every object A there is associated an arrow id_A, and with every pair of compatible arrows $f \colon A \to B$, $g \colon B \to C$ there is associated an arrow $g \circ f \colon A \to C$ — that is, a deductive system is a collection of proofs (deductions) between the formulae together with at least one axiom $\frac{}{A \xrightarrow{id_A} A}$ (identity), and at least one rule $\frac{A \xrightarrow{f} B \quad B \xrightarrow{g} C}{A \xrightarrow{g \circ f} C}$ (cut). A category is a deductive system satisfying the obvious coherence conditions — for all compatible arrows: $A \xrightarrow{f} B$, $B \xrightarrow{g} C$, $C \xrightarrow{h} D$ the following holds: $h \circ (g \circ f) = (h \circ g) \circ f$ and $f \circ id_A = f = id_B \circ f$.

[1] And the Heyting algebra consisting of the open subsets of the real line is complete for intuitionistic logic.