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Harish-Chandra invariants and the centre of the reduced enveloping algebra

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ABSTRACT

In this article we consider the centre of the reduced enveloping algebra of the Lie algebra of a reductive algebraic group in very good characteristic $p > 2$. The Harish-Chandra centre maps to the centre of each reduced enveloping algebra and, using a combination of induction and deformation arguments, we describe precisely for which p-characters this map is surjective: it is if and only if the chosen character is regular. This provides the converse to a theorem of Mirković and Rumynin.
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1. Introduction

Throughout we take $\Bbbk$ to be an algebraically closed field of characteristic $p > 2$ and G a connected reductive algebraic group of rank ℓ with simply connected derived subgroup. We also assume that p is very good for G. The Lie algebra $\mathfrak{g}$ is restricted in a natural way and the representation theory of $\mathfrak{g}$ is governed by that of the reduced enveloping algebras $U_\chi(\mathfrak{g})$ with $\chi \in \mathfrak{g}^*$. When the stabiliser of χ in $\mathfrak{g}$ is a torus the reduced enveloping algebra is semisimple whilst, at the other end of the spectrum, the simple $U_0(\mathfrak{g})$-modules are precisely the differentials of simple G-modules with restricted highest weights.

Since $U_\chi(\mathfrak{g})$ plays a key role in the representation theory of $\mathfrak{g}$ it seems important to understand the centre $Z_\chi(\mathfrak{g}) := Z(U_\chi(\mathfrak{g}))$ and here we make some progress towards that end. For $\chi \in \mathfrak{g}^*$ we consider the map

$$\varphi_\chi : Z(\mathfrak{g}) \longrightarrow Z_\chi(\mathfrak{g})$$

obtained by restricting the projection $U(\mathfrak{g}) \twoheadrightarrow U_\chi(\mathfrak{g})$. Recall that the centre $Z(\mathfrak{g})$ of the enveloping algebra $U(\mathfrak{g})$ is well understood, thanks to the work of several authors. We know that $Z(\mathfrak{g})$ is generated by the p-centre $Z_p(\mathfrak{g})$ and the Harish-Chandra invariants $U(\mathfrak{g})^G$, and the latter has a similar description in characteristic zero and very good positive characteristic. This leads immediately to the question, when is φ_χ

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surjective? The two extremal cases of this question are known: when χ is regular a theorem of Mirković and Rumynin says that φ_χ surjects [11, Theorem 12], whilst in the case $\chi = 0$ the map φ_χ does not surject, by an argument due to Premet [2, §3.17]. Apart from this very little is known about the cokernel of φ_χ as a linear map, and the goal of this article is to describe precisely when it is zero.

We remind the reader that an element $\chi \in \mathfrak{g}^*$ is called regular if the stabiliser $\mathfrak{g}_\chi := \{x \in \mathfrak{g} : \chi[x, \mathfrak{g}] = 0\}$ has the minimal possible dimension $\text{rank}(\mathfrak{g})$. The following is our main result.

Theorem. $Z_\chi(\mathfrak{g}) = \varphi_\chi(Z(\mathfrak{g}))$ *if and only if* χ *is a regular element.*

The idea behind the proof is quite simple. To start with we observe that isomorphism type and hence the dimension of the centre of $U_\chi(\mathfrak{g})$ only depends upon the orbit $\text{Ad}^*(G)\chi$. In the base case $G = SL_2$ or $\mathbb{G}_m$ and coadjoint orbits are either regular or trivial. In these cases the theorem follows from previously established results. When $\ell > 1$, once again the regular orbits are dealt with by [11], whilst every non-regular orbit $\mathcal{O} \subseteq \mathfrak{g}^*$ lies in the closure of a non-nilpotent subregular decomposition class. If χ lies in such a decomposition class we show that $Z_\chi(\mathfrak{g}) \cong Z_\chi(\mathfrak{g}_{\chi_s})$ for $\chi \in \mathcal{O}$ and apply an inductive argument, whilst if χ lies in the boundary of this decomposition class we apply a deformation argument to show that the dimension of the centre is larger than $\dim \varphi_\chi(Z(\mathfrak{g}))$.

The paper is organised as follows. In Section 2 we fix our notation and recall the elements of the theory. We then explain that $\dim \varphi_\chi(Z(\mathfrak{g})) = p^\ell$ and go on to recall an important category equivalence due to Kac–Weisfeiler which allows us to relate the centre $Z_\chi(\mathfrak{g})$ to $Z_\chi(\mathfrak{g}_{\chi_s})$. In Section 3 we recall the two known cases of the theorem and in Section 4 we recall the theory of decomposition classes. It is here that we show that $\dim Z_\chi(\mathfrak{g})$ does not change as we vary χ over such a class and prove a useful result which states that $\mathfrak{g}^* \to \mathbb{Z}_{\geqslant 0}; \chi \mapsto \dim(Z_\chi(\mathfrak{g}))$ is upper semicontinuous. In the final section we combine the ingredients and complete the proof using induction on the reductive rank of G.

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2. Notations and preliminaries

Our notations and assumptions are the following:

(1) G is a reductive algebraic group over $\Bbbk$;
(2) $\Bbbk$ is algebraically closed of characteristic $p > 2$ and p is very good for G;
(3) G is connected and the derived subgroup $[G, G]$ is simply connected.

We warn the reader that not every Levi subgroup satisfies these same hypotheses. In particular p will not necessarily be a very good prime for every Levi subgroup. Nonetheless, Lie algebras of Levi subgroups are precisely the centralisers of semisimple elements (see [8, 2.2(4)] and [3, Lemma 2.1.2]).

We always write $\mathfrak{g} = \text{Lie}(G)$, write $U(\mathfrak{g})$ for the enveloping algebra and $S(\mathfrak{g})$ for the symmetric algebra. We shall denote the centre of $\mathfrak{g}$ by $\mathfrak{z}(\mathfrak{g})$ and we observe that, since the characteristic of $\Bbbk$ is very good for G, we have $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \bigoplus [\mathfrak{g}, \mathfrak{g}]$ and, more generally, $\mathfrak{l} = \mathfrak{z}(\mathfrak{l}) \bigoplus [\mathfrak{l}, \mathfrak{l}]$ for every Levi subalgebra $\mathfrak{l} = \text{Lie}(L) \subseteq \mathfrak{g}$ such that p is very good for L (see [15, §2.1] for slightly more detail).

The maximal ideal of the p-centre corresponding to $\chi \in \mathfrak{g}^*$ shall be written $I_\chi := (x^p - x^{[p]} - \chi(x)^p : x \in \mathfrak{g})$. Then the reduced enveloping algebra is

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