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Modules over quantaloids: Applications to the isomorphism problem in algebraic logic and π-institutions

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A B S T R A C T

We solve the isomorphism problem in the context of abstract algebraic logic and of π-institutions, namely the problem of when the notions of syntactic and semantic equivalence among logics coincide. The problem is solved in the general setting of categories of modules over quantaloids. We introduce closure operators on modules over quantaloids and their associated morphisms. We show that, up to isomorphism, epis are morphisms associated with closure operators. The notions of (semi-)interpretability and (semi-)representability are introduced and studied. We introduce cyclic modules, and provide a characterization for cyclic projective modules as those having a g-variable. Finally, we explain how every π-institution induces a module over a quantaloid, and thus the theory of modules over quantaloids can be considered as an abstraction of the theory of π-institutions.

1. Introduction

In order to study the property of algebraizability for sentential logics, and the equivalence between deductive systems in general, Blok and Jónsson introduced the notion of *equivalence* between *structural closure operators* on a set X acted on by a monoid M, or an *M-set* (see [2]). As usual, given a monoid $\langle M, \cdot, 1\rangle$, an M-set consists of a set X and a monoid action $\star : M \times X \to X$, where $1 \star x = x$ and $a \star (b \star x) = (a \cdot b) \star x$, for all $a, b \in M$ and $x \in X$. While the use of closure operators to encode entailment relations is very well known, the action of the monoid is introduced to formalize the notion of structurality, that is, "entailments are preserved by uniform substitutions," a property usually required for logics.

Given an M-set $\langle X, \star\rangle$, a closure operator C on X is *structural* on $\langle X, \star\rangle$ if and only if it satisfies the following property: for every $\sigma \in M$, and every $\Gamma \subseteq X$, $\sigma \star C\Gamma \subseteq C(\sigma \star \Gamma)$, where $\sigma \star \Gamma = \{\sigma \star \varphi : \varphi \in \Gamma\}$. This can be shortly written as follows:

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$$\forall \sigma \in M, \quad \sigma C \leqslant C\sigma. \tag{Str}$$

This is known as the *structurality property* for C, since it takes the following form, when expressed in terms of $\vdash_C$, the closure relation on X associated with the closure operator C (defined by $\Gamma \vdash_C \varphi$ if and only if $\varphi \in C\Gamma$): for every $\Gamma \subseteq X$, every $\varphi \in X$, and every $\sigma \in M$,

$$\Gamma \vdash_C \varphi \;\Rightarrow\; \sigma \star \Gamma \vdash_C \sigma \star \varphi.$$

For every $\sigma \in M$, a unary operation $C\sigma$ on $\mathbf{Cl}(C)$, the lattice of *theories* or *closed sets* of C, is defined in the following way: $C\sigma(\Gamma) = C(\sigma \star \Gamma)$. The *expanded lattice of theories* of a structural closure operator C is defined as the structure $\langle \mathbf{Cl}(C), (C\sigma)_{\sigma \in M} \rangle$.

In their approximation, Blok and Jónsson define two structural closure operators on two M-sets to be equivalent if their expanded lattices of theories are isomorphic. Later, they prove that under certain hypotheses (the existence of basis), this is equivalent to the existence of conservative and mutually inverse interpretations, which is the original idea of equivalence between deductive systems emerging from the work of Blok and Pigozzi. This equivalence between the lattice-theoretic property of having isomorphic expanded lattices of theories, and the semantic property of being mutually interpretable is known by the name of the *Isomorphism Theorem*. And the problem of determining in which situations there exists an Isomorphism Theorem is called the *Isomorphism Problem*.

The first Isomorphism Theorem was proved by Blok and Pigozzi in [3] for algebraizable sentential logics, and later it was obtained for k-dimensional deductive systems by them in [4] and for Gentzen systems by Rebagliato and Verdú in [18]. But there is not a general Isomorphism Theorem for structural closure operators on M-sets, as there are counterexamples for that (see [9]).

In turn, Voutsadakis studied in [21] the notion of equivalence of π-institutions at different levels (quasi-equivalence and deductive equivalence) and identified term π-institutions, for which a certain kind of Isomorphism Theorem also holds. The notion of π-institution was introduced by Fiadeiro and Sernadas in their article [7] and can be viewed as a generalization of deductive systems allowing multiple sorts. They constitute a very wide categorical framework embracing sentential logics, Gentzen systems, etc., as they include structural closure operators on M-sets as a particular case. Therefore, a general Isomorphism Theorem for π-institutions is not possible (see [8]).

Sufficient conditions for the existence of an Isomorphism Theorem were provided in [9] and [8] for structural closure operators on M-sets (and graded M-sets), and π-institutions that encompass all the previous known cases. The first complete solution of the Isomorphism Problem was found for closure operators on modules over residuated complete lattices, or *quantales* (see [10]). In this article, the modules providing an Isomorphism Theorem are identified as the projective modules. In particular, cyclic projective modules are characterized in several ways, from which the Isomorphism Theorem for k-deductive systems follows, and also for Gentzen systems, using that coproducts of projectives are projective. The Isomorphism Problem for π-institutions remained open.

In this article we present a solution for the Isomorphism Problem in the general framework of closure operators on modules over quantaloids, and as a particular case for π-institutions. In order to do that, the theory of modules over quantaloids and of closure operators on them is developed in a categorical way.

We introduce the notions of (semi-)interpretability and (semi-)representability of one closure operator into another and study their relationships. We prove that the set of closure operators that are interpretable by a given morphism τ is a principal filter of the lattice of closure operators on its domain. As a consequence, we obtain that every extension of an interpretable closure operator is also interpretable by the same morphism. One instantiation of this result is the well-known fact that if a sentential logic has an algebraic semantics, then every extension of it also has an algebraic semantics and with the same defining equations. This is the content of Theorem 2.15 of [5].